Subspaces

**Definition:** A nonempty subset $S$ in $\mathbb{R}^n$ is called a **subspace** if and only if the following properties are satisfied:

1) If $\vec{u}$ and $\vec{v}$ are in $S$, then $\vec{u} + \vec{v}$ is in $S$. That is, $S$ is **closed under vector addition**. (VA)
2) If $\vec{u}$ is in $S$ and if $c$ is any scalar, then $c\vec{u}$ is also in $S$. That is, $S$ is **closed under scalar multiplication**. (SM)

In other words, if $S$ is a subspace, then any linear combination of vectors in $S$ will remain in $S$. Also, because the constants in the linear combination can always be 0, **EVERY** subspace must contain $\vec{0}$.

**Theorem:** A nonempty subset $S$ of $\mathbb{R}^n$ is a subspace if and only if it is the span of some finite set of vectors $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_k\}$ in $\mathbb{R}^n$. (That is, if it’s a subspace, it’s a span, and if it’s a span, it’s a subspace. It goes both ways!)

### Recognizing Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

<table>
<thead>
<tr>
<th>Subspaces of $\mathbb{R}^2$ : ${(x, y)}$</th>
<th>Subspaces of $\mathbb{R}^3$ : ${(x, y, z)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ (0,0) }$ ONLY if both $x$ and $y$ are 0</td>
<td>${ (0,0,0) }$ ONLY if $x$, $y$, and $z$ are all 0</td>
</tr>
<tr>
<td><strong>Line through (0,0)</strong></td>
<td><strong>Line through (0,0,0)</strong></td>
</tr>
</tbody>
</table>
| Slope-Intercept Form: $y = mx$ | Parametric Form:  
$\begin{cases}  
  x = x_0 + ta \\
  y = y_0 + tb \\
  z = z_0 + tc
\end{cases}$ |
| General Form: $ax + by = 0$ | Vector Form:  
$\begin{bmatrix}  
  x \\
  y \\
  z
\end{bmatrix} =  
\begin{bmatrix}  
  x_0 \\
  y_0 \\
  z_0
\end{bmatrix} + t  
\begin{bmatrix}  
  a \\
  b \\
  c
\end{bmatrix} + s  
\begin{bmatrix}  
  a_2 \\
  b_2 \\
  c_2
\end{bmatrix}$ |
| Parametric Form:  
$\begin{cases}  
  x = x_0 + ta \\
  y = y_0 + tb
\end{cases}$ | Parametric Form:  
$\begin{cases}  
  x = x_0 + ta_1 + sa_2 \\
  y = y_0 + tb_1 + sb_2 \\
  z = z_0 + tc_1 + sc_2
\end{cases}$ |
| Vector Form:  
$\begin{bmatrix}  
  x \\
  y
\end{bmatrix} =  
\begin{bmatrix}  
  x_0 \\
  y_0
\end{bmatrix} + t  
\begin{bmatrix}  
  a \\
  b
\end{bmatrix}$ | Vector Form:  
$\begin{bmatrix}  
  x \\
  y \\
  z
\end{bmatrix} =  
\begin{bmatrix}  
  x_0 \\
  y_0 \\
  z_0
\end{bmatrix} + t  
\begin{bmatrix}  
  a_1 \\
  b_1 \\
  c_1
\end{bmatrix} + s  
\begin{bmatrix}  
  a_2 \\
  b_2 \\
  c_2
\end{bmatrix}$ |
| $\mathbb{R}^2$ - every single vector of the form $(x, y)$ | $\mathbb{R}^3$ - every single vector of the form $(x, y, z)$ |
Methods for Determining if something is a Subspace

Method 1:
1. Check to see if \( \vec{0} \in S \). If it is not, \( S \) is not a subspace, and you are done.

2. Write two generic vectors in the space, \( \vec{v}_1 = (x_1, y_1, z_1) \) and \( \vec{v}_2 = (x_2, y_2, z_2) \).

3. Show that Vector Addition (VA) holds - add the two vectors together and show that they still maintain the properties necessary to be in \( S \).

4. Show that Scalar Multiplication holds (SM) - multiply one of the generic vectors by a scalar \( k \) and show that it still maintains the properties necessary to be in \( S \).

5. If either VA or SM fails, \( S \) is not a subspace, and you must provide a counter example.

Method 2:
1. Check to see if \( \vec{0} \in S \). If it is not, \( S \) is not a subspace, and you are done.

2. Check to see if \( S \) is the span of some vectors in \( \mathbb{R}^n \) - using the above chart will be helpful for this.

3. If yes to both, \( S \) is a subspace. If no to even one of the questions, \( S \) is not a subspace, and you must provide a counterexample.

Note: Sometimes it is not obvious whether or not \( S \) can be written as a span of vectors. In this case, you must use Method 1.

Example: Show that each of the following satisfies the closure properties and is therefore a subspace of \( \mathbb{R}^n \). Also, write them as the span of a finite set of vectors.

a) \( S = \{ (x, y, z) \in \mathbb{R}^3 | x = 4t, y = -2t, z = t \} \)

Method 1:
1. If \( t = 0 \) then \( (x, y, z) = (0, 0, 0) \), so \( \vec{0} \in S \).

2. \( \vec{v}_1 = (x_1, y_1, z_1) \) where \( x_1 = 4t_1, y_1 = -2t_1, z_1 = t_1 \) and \( \vec{v}_2 = (x_2, y_2, z_2) \) where \( x_2 = 4t_2, y_2 = -2t_2, z_2 = t_2 \)

3. VA: \( \vec{v}_1 + \vec{v}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \) We need \( x_1 + x_2 = 4t_1 + 4t_2 = 4(t_1 + t_2), y_1 + y_2 = -2t_1 - 2t_2 = -2(t_1 + t_2), \) and \( z_1 + z_2 = t_1 + t_2 \)
   We can let \( t = t_1 + t_2 \), and we see that vector addition holds.

4. SM: \( k\vec{v} = (kx_1, ky_2, kz_3) \) We need \( kx_1 = 4t, ky_1 = -2y, \) and \( kz_1 = t \)
   \( kx_1 = k(2t_1) = 2(kt_1), ky_1 = k(-2t_1) = -2(kt_1), \) and \( kz_1 = kt_1 \)
   We can let \( t = kt_1 \) and scalar multiplication holds.

So \( S \) is a subspace. Now we want to write it as a span of vectors. We know that \( \left\{ \begin{array}{l} x = 4t \\ y = -2t \\ z = t \end{array} \right. \) is the parametric equation for a line going through \( (0, 0, 0) \) with parallel vector \( (4, -2, 1) \). So this subspace equals \( \text{span}\{(4, -2, 1)\} \).
b) $S = \{(x, y, z) \in \mathbb{R}^3 | 3x - 4y + 5z = 0\}$

**Method 1:**
1. $(0, 0, 0)$ satisfies $3x - 4y + 5z = 0$, so $\vec{0} \in S$
2. $\vec{v}_1 = (x_1, y_1, z_1)$ where $3x_1 - 4y_1 + 5z_1 = 0$ and $\vec{v}_2 = (x_2, y_2, z_2)$ where $3x_2 - 4y_2 + 5z_2 = 0$
3. $\vec{v}_1 + \vec{v}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ We need $3(x_1 + x_2) - 4(y_1 + y_2) + 5(z_1 + z_2) = 0$
   $= 3x_1 + 3x_2 - 4y_1 - 4y_2 + 5z_1 + 5z_2 = 3x_1 - 4y_1 + 5z_1 + 3x_2 - 4y_2 + 5z_2 = 0 + 0 = 0$
   We can see that vector addition holds.
4. $k\vec{v}_1 = (kx_1, ky_1, kz_1)$ $3kx_1 - 4ky_1 + k5z_1 = k(3x_1 - 4y_1 + 5z_1) = 0$ We can see that scalar multiplication holds.

So $S$ is a subspace. We must now find the span, and we can do this by setting up a matrix and finding the solution space.

$$\begin{bmatrix} 3 & -4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

So $y$ and $z$ are free variables, and we can see that

$$\begin{cases} x = 4/3t - 5/3s \\ y = t \\ z = s \end{cases}$$

so that

$$S = \text{span}\{(4/3, 1, 0), (-5/3, 0, 1)\}.$$

**Note:** The zero subspace $\{\vec{0}\}$ and $\mathbb{R}^n$ itself are always subspaces of $\mathbb{R}^n$. 