APPLICATIONS OF DETERMINANTS

ADJOINTS

The **adjoint** of a matrix A, denoted adj(A), is defined to be the transpose of the cofactor matrix. That is, adj$(A) = [C_{ij}]^T$.

Steps for finding the adjoint Matrix:

Step 1: First, find the minor matrix.

Step 2: Second, find the cofactor matrix. Remember that the NUMBERS in the cofactor matrix are the same as the numbers in the minor matrix. The only thing that may or may not be different is the sign.

Step 3: Take the transpose of the cofactor matrix.

Example: Find the adjoint of matrix $A = \begin{bmatrix} 2 & 1 & -3 \\ 6 & 4 & -1 \\ 5 & -2 & 3 \end{bmatrix}$:

Step 1: $M = \begin{bmatrix} 10 & 23 & -32 \\ -3 & 21 & -9 \\ 11 & 16 & 2 \end{bmatrix}$

Step 2: $C = \begin{bmatrix} 10 & -23 & -32 \\ 3 & 21 & 9 \\ 11 & -16 & 2 \end{bmatrix}$

Step 3: adj$(A) = \begin{bmatrix} 10 & 3 & 11 \\ -23 & 21 & -6 \\ -32 & 9 & 2 \end{bmatrix}$

The adjoint is useful because it gives us another way to solve for the inverse of a matrix.

Let A be an $n \times n$ square matrix. Then

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

Example: Find the inverse of the above matrix, A, by using the adjoint formula.

We have already calculate the adjoint, so we need only calculate the determinant. This should be easy to do using cofactor expansion, as we already have all of the cofactors. Let's expand down the second column.

$\det(A) = (1)(-23) + (4)(21) + (-2)(-16) = 93$

Therefore $A^{-1} = \frac{1}{93}\begin{bmatrix} 10 & 3 & 11 \\ -23 & 21 & -6 \\ -32 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 10/93 & 3/93 & 11/93 \\ -23/93 & 21/93 & -6/93 \\ -32/93 & 9/93 & 2/93 \end{bmatrix}$

Cramer's Rule

The determinant can also be useful in solving systems of equations. To solve systems of equations with determinants, we use a method called Cramer's Rule.

Cramer's Rule:
If A is an $n \times n$ matrix with $\det(A) \neq 0$, then the linear system $Ax = B$ has the unique solution $X = (x_j)$ given by $x_j = \dfrac{\det(A_j)}{\det(A)}$ for $j = 1, 2, ..., n$ where A_j is the matrix obtained by replacing the jth column of A by B.

Steps for Using Cramer's Rule:

Step 1: Set up the system $AX = B$
Step 2: Calculate the determinant of A
Step 3: To solve for a variable, write the matrix A_j by replacing the jth column, the one associated with the variable, with the matrix B, and calculate the determinant. Do this for each variable.
Step 4: Plug in to the formula to solve for the in question.

Example: Solve the system $\begin{cases} 2x - y + 5z = 6 \\ x - 3y + 4z = 2 \\ 3x + 3y + z = 3 \end{cases}$ using Cramer's Rule.

Step 1: $$\underset{A}{\begin{bmatrix} 2 & -1 & 5 \\ 1 & -3 & 4 \\ 3 & 3 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}}$$

Step 2: To calculate the determinant, I will expand down the first column.
$M_{11} = \begin{vmatrix} -3 & 4 \\ 3 & 1 \end{vmatrix} = -15, M_{21} = \begin{vmatrix} -1 & 5 \\ 3 & 1 \end{vmatrix} = -16$, and $M_{31} = \begin{vmatrix} -1 & 5 \\ -3 & 4 \end{vmatrix} = 11$
From this we get, $C_{11} = -13, C_{21} = 16$, and $C_{31} = 11$, giving us
$\det(A) = (2)(-15) + (1)(16) + (3)(11) = 19$

Step 3: a) To solve for x, replace the first column with B and calculate the determinant:
$$\det(A_1) = \begin{vmatrix} 6 & -1 & 5 \\ 2 & -3 & 4 \\ 3 & 3 & 1 \end{vmatrix} = (6)(-15) + (2)(16) + (3)(11) = -25$$
b) To solve for y, replace the second column with B and calculate the determinant:
$$\det(A_2) = \begin{vmatrix} 2 & 6 & 5 \\ 1 & 2 & 4 \\ 3 & 3 & 1 \end{vmatrix} = (2)(-10) - (1)(-9) + (3)(14) = 31$$
c) To solve for z, replace the third column with B and calculate the determinant:
$$\det(A_3) = \begin{vmatrix} 2 & -1 & 6 \\ 1 & -3 & 2 \\ 3 & 3 & 3 \end{vmatrix} = (2)(-15) - (1)(-21) + (3)(16) = 39$$

Step 4: $x = \dfrac{-25}{19}, y = \dfrac{31}{19}, z = \dfrac{39}{19}$