DETERMINANTS

The **determinant** is a function applied to **square** matrices that has no analogy in real numbers. The matrix A is invertible (non-singular) if and only iff $det(\text{A}) \neq 0$. We write the determinant of a matrix A as $\det(A)$ or $|A|$.

How to compute the determinant of A

When A is a 2×2 matrix:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = |A| = ad - bc$

Definition: If A is a square matrix, then its (i, j) **minor**, denoted by M_{ij}, is the determinant of the matrix formed by eliminating the ith row and the jth column from A.

Example: Let $A = \begin{bmatrix} 7 & 6 & -3 \\ 2 & 1 & 5 \\ 4 & 8 & 9 \end{bmatrix}$

Then $M_{32} = \begin{vmatrix} 7 & 6 & -3 \\ 2 & 1 & 5 \\ 4 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 2 & 5 \end{vmatrix} = (7)(5) - (2)(-3) = 35 + 6 = 41$

Definition: The **cofactor** of entry a_{ij} is denoted by C_{ij} and is the number $(-1)^{i+j} M_{ij}$

Example: Given the matrix A above, $C_{32} = (-1)^{3+2}(41) = -41$

The cofactor is therefore just the signed minor. The following matrix may be helpful in remembering the sign of a cofactor: $\begin{bmatrix} + & - & + & - & + & \ldots \\ - & + & - & + & - & \ldots \\ + & - & + & - & + & \ldots \\ - & + & - & + & - & \ldots \\ & \ldots & & & \ldots & \end{bmatrix}$

The determinant of a matrix can be calculated by **cofactor expansion** along any row or column:

Cofactor Expansion across the i**th** row: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$

Cofactor Expansion down the j**th** column: $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$

Example: Find the determinant of $A = \begin{bmatrix} 7 & 6 & -3 \\ 2 & 1 & 5 \\ 4 & 8 & 9 \end{bmatrix}$ by cofactor expansion down the 2nd column.

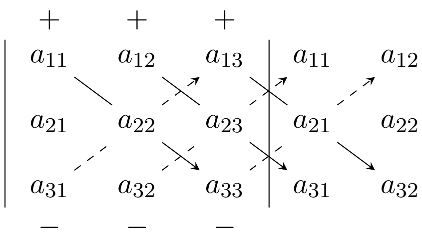
$\det(A) = (6)(-1)^{1+2} \begin{vmatrix} 2 & 5 \\ 4 & 9 \end{vmatrix} + (1)(-1)^{2+2} \begin{vmatrix} 7 & -3 \\ 4 & 9 \end{vmatrix} + (8)(-1)^{3+2} \begin{vmatrix} 7 & -3 \\ 2 & 5 \end{vmatrix} =$

$(-6)[(9)(2) - (5)(4)] + (1)[(7)(9) - (-3)(4)] + (-8)[(7)(5) - (-3)(2)] =$

$(-6)(18 - 20) + (1)(63 + 12) + (-8)(35 + 6) = (-6)(-2) + (1)(75) + (-8)(41) = -241$

When A is a 3×3 matrix:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\det(A)$ can also be computed by using **Sarrus's rule**.



Add together the products from the downward arrows then subtract the products from the upwards dashed arrows.
$\det(A) = |A| = [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}]$

It is often simpler to first perform row operations before calculating the determinant. Let's see why and understand **how row operations applied to a matrix affect its determinant**.

Row and Column Operations

Theorem: Let A be an $n \times n$ square matrix. Then the value of $\det(A)$ is affected by the elementary row operations as follows:

i. If A_1 is obtained by interchanging two rows, then $\det(A_1) = -\det(A)$
ii. If A_2 is obtained by multiplying a row by the non-zero constant k, then $\det(A_2) = k\det(A)$
iii. If A_3 is obtained by the addition of a multiple of one row to another, then $\det(A_3) = \det(A)$. That is, the determinant is unchanged!!

Column Operations: Note that when calculating the determinant, any row operation can also be done as a **column operation**, by replacing the word row with the word column above. These column operations have the same effect as the equivalent row operations.

Properties of the Determinant

Let A be an $n \times n$ matrix. Then

(1) If A contains a row or column of 0s, then $\det(A)=0$
(2) $\det(A^T) = \det(A)$
(3) $\det(I) = 1$
(4) If A is triangular then det(A)=the elements of the diagonal multiplied together
(5) If A contains two proportional rows (or columns), then $\det(A) = 0$
(6) $\det(kA) = k^n \det(A)$
(7) $\det(AB) = \det(A)\det(B)$
(8) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Equivalent Statements: If A is an $n \times n$ matrix, the following assertions are equivalent (if one is true, all are true!)

a) A is invertible
b) $AX = 0$ has only the trivial (0) solution
c) A reduces to I
d) $AX = B$ is consistent for EVERY $n \times 1$ matrix B
e) $AX = B$ has a unique solution for every $n \times 1$ matrix B.
f) $\det(A) \neq 0$