Equations of Planes and Lines

Remember that the **general equation** of a plane in $\mathbb{R}^3$ is $ax + by + cz + d = 0$ and the general equation for a line in $\mathbb{R}^2$ is $ax + by + c = 0$.

Point Normal Equations

A line in $\mathbb{R}^2$ containing a point $P(x_0, y_0)$ with normal $\vec{n} = (a, b)$ can be expressed: $a(x - x_0) + b(y - y_0) = 0$.

A plane in $\mathbb{R}^3$ containing a point (x_0, y_0, z_0) with normal $\vec{n} = (a, b, c)$ can be expressed: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Theorem:
a) If a and b are constants that are not both 0, then an equation of the form $ax + by + c = 0$ represents a line in $\mathbb{R}^2$ with normal $\vec{n} = (a, b)$.

b)If a, b, and c are constant that are not all 0, then an equation of the form $ax + by + cz + d = 0$ represents a plane in $\mathbb{R}^3$ with normal $\vec{n} = (a, b, c)$.

Example: Find a vector orthogonal to the line $2x + 6y - 3 = 0$.
We know that the normal vector represents a vector orthogonal to the line, and we can see that the normal is $\vec{n} = (2, 6)$.

Example: Find a vector orthogonal to the plane $7x + 6y - z = 0$.
We know that the normal vector represents a vector orthogonal to the plane, and we can see that the normal is $\vec{n} = (7, 6, -1)$.

Theorem:
a) If the normal vectors of two lines are multiples of each other, then those lines are either parallel or the same line.
b) If the normal vectors of two lines/planes are orthogonal (with dot product=0), then those lines/planes must also be orthogonal.

Vector Equations

A line in $\mathbb{R}^3$ can be written in vector form as: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
where $P_0 = (x_0, x_0, z_0)$ is a point on the line and $\vec{v} = (a, b, c)$ is a vector **parallel** to the line.

A plane in $\mathbb{R}^3$ can be written in vector form as: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + s \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$
where $P_0 = (x_0, y_0, z_0)$ is a point on the plane and $\vec{v_1} = (a_1, b_1, c_1)$ and $\vec{v_2} = (a_2, b_2, c_2)$ are vectors **parallel** to the plane.

To find the vector equation of a line in $\mathbb{R}^3$ through two points, P and Q

Step 1: Find the vector $\overrightarrow{PQ}$. This vector will be parallel to the line.

Step 2: Set up the vector equation as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P + t\overrightarrow{PQ}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q + t\overrightarrow{PQ}$

To find the general equation of a plane in $\mathbb{R}^3$ through three points, P, Q, and R

Step 1: Find two vectors on the plane, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (Note that you could also use $\overrightarrow{QR}$

Step 2: Take the cross product of the two vectors to find a vector normal to the plane: $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

Step 3: Now that you have a normal, plug the normal along with a point into the point-normal equation (it doesn't matter which point you use).

To find the vector equation of a plane in $\mathbb{R}^3$ through three points, P, Q, and R

Step 1: Find two vectors on the plane, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (Note that you could also use $\overrightarrow{QR}$

Step 2: As these two vectors are on the plane, they must be parallel to the plane, so use them to set up the vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P + t\overrightarrow{PQ} + s\overrightarrow{PR}$.

Note that you could choose P, Q, or R as your point.