

Intro to Vectors

A **vector** is an ordered set of numbers that represents length (magnitude) and direction.

Notation: A vector is typically denoted either in bold, \mathbf{v} , or with an arrow above it \vec{v} .

We usually use the same letter for the different components of a vector. So a vector in 3-space with 3 components might be written $\vec{v} = \langle v_1, v_2, v_3 \rangle$ or $\vec{v} = (v_1, v_2, v_3)$. The $\langle \rangle$ and $()$ brackets are used interchangeably. I can also use square brackets, $[\]$, and write them as column matrices.

When considering a vector $\vec{u} = (u_1, u_2)$ in \mathbb{R}^2 , we must understand that u_1 represents the displacement in the x-direction and u_2 represents the displacement in the y-direction - it does NOT represent a location; just a length and a direction.

Equality of Vectors: Two vectors are said to be **equal** if their components are equal; that is, if they have the same direction and magnitude.

If we want for a vector to be located in a specific place, it is necessary to give the vector an **initial point**, which indicates where the vector will start. In this case, the point where it ends will be called the **terminal point**.

A vector between two points: If $P_1(x_1, y_1)$ is the initial point and $P_2 = (x_2, y_2)$ is the terminal point for the vector \vec{u} , then $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$

If a vector is placed so that its initial point is the origin, O , with a terminal point of P , then it is called a **position vector**. Unless we are specifically given initial points or terminal points, we will think of any vector in \mathbb{R}^n is a position vector.

Vector Operations

Addition/Subtraction of Vectors: If $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$ are vectors then $\vec{u} \pm \vec{v} = (u_1 \pm v_1, u_2 \pm v_2, \dots, u_n \pm v_n)$

This means to first do the length and direction of \vec{u} and then do the length and direction of \vec{v} (or vice versa).

Scalar Multiplication of Vectors: If $\vec{v} = (v_1, v_2, \dots, v_n)$ is a vector and k is a scalar, then $k\vec{v} = (kv_1, kv_2, \dots, kv_n)$

Scalar multiplication of a vector does the following:

- 1) Stretches it (makes it longer) if $k > 1$
- 2) Shrinks it (makes it shorter) if $0 < k < 1$
- 3) Changes its direction if $k < 0$

Definition: Two vectors are said to be **parallel** if they point in the same or opposite directions (if the angle between them is 0)

Example: The vector $\vec{u} = (3, 5, -2, 4)$ is parallel to the vector $\vec{v} = (-6, -10, 4, -8)$ because they point in opposite directions. This can be seen because \vec{u} is a scalar multiple of \vec{v} .

Properties of Vectors

Let \vec{u}, \vec{v} , and \vec{w} be vectors in \mathbb{R}^n , and let a and b be scalars. Then:

- a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- c) $a(b\vec{u}) = (ab)\vec{u}$
- d) $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
- e) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- f) $\vec{u} + \vec{0} = \vec{u}$
- g) $1(\vec{u}) = \vec{u}$
- h) $a(\vec{0}) = \vec{0}$, and $0(\vec{u}) = \vec{0}$
- i) $\vec{u} + (-\vec{u}) = \vec{0}$
- j) If $a\vec{u} = \vec{0}$, then either $a = 0$ or $\vec{u} = \vec{0}$

The Magnitude (or Norm) of a Vector

The **magnitude** of a vector $\vec{v} = (v_1, v_2, \dots, v_n)$ is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

Definition: A vector of norm 1 is called a **unit vector**.

Given a non-zero vector \vec{v} in \mathbb{R}^n , $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector in the same direction as \vec{v} .

Theorem: If \vec{v} is any non-zero vector in \mathbb{R}^n , then

- a) $\|\vec{v}\| \geq 0$
- b) $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$
- c) $\|k\vec{v}\| = |k| \|\vec{v}\|$

The Dot Product

If $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$ are vectors in \mathbb{R}^n then the **dot product** of \vec{u} and \vec{v} is given by $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$.

Example: Find the dot product of $\vec{u} = (1, 3, 2, 0)$ and $\vec{v} = (6, -3, 1, 2)$.

$$\vec{u} \cdot \vec{v} = (1)(6) + (3)(-3) + (2)(1) + (0)(2) = 6 - 9 + 2 = -1$$

Theorem: Two non-zero vectors \vec{u} and \vec{v} are said to be **orthogonal** (or perpendicular) if $\vec{u} \cdot \vec{v} = 0$. We write $\vec{u} \perp \vec{v}$

Definition: A non-empty set of vectors in \mathbb{R}^n is said to be an **orthogonal set** if ALL pairs of vectors in the set are orthogonal.

Example: Do the vectors $\vec{u} = (1, 2, 3)$, $\vec{v} = (-4, 5, -2)$, and $\vec{w} = (3, 6, -5)$ form an orthogonal set?

Check each pair separately:

$$\vec{u} \cdot \vec{v} = (1)(-4) + (2)(5) + (3)(-2) = -4 + 10 - 6 = 0, \text{ so } \vec{u} \perp \vec{v}$$

$$\vec{u} \cdot \vec{w} = (1)(3) + (2)(6) + (3)(-5) = 3 + 12 - 15 = 0, \text{ so } \vec{u} \perp \vec{w}$$

$\vec{v} \cdot \vec{w} = (-4)(3) + (5)(6) + (-2)(-5) = -12 + 30 + 10 = 28 \neq 0$, so \vec{v} is not perpendicular to \vec{w} . Because \vec{v} and \vec{w} are not orthogonal, these vectors do not form an orthogonal set.

The Cross Product

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in \mathbb{R}^3 then the **cross product** of \vec{u} and \vec{v} is

$$\text{given by } \vec{u} \times \vec{v} = \left(\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \right)$$

Example: Find the cross product of $\vec{u} = (1, 2, 3)$ and $\vec{v} = (-2, 1, 4)$

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}, - \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \right) = (5, -10, 5)$$

Theorem: If \vec{u}, \vec{v} are two non-parallel non-zero vectors in \mathbb{R}^3 , then $\vec{w} = \vec{u} \times \vec{v}$ is orthogonal to both \vec{u} and \vec{v} .