Linear Combinations and Spans

Linear Combinations

Definition: If $\vec{u_1}, \vec{u_2}, ..., \vec{u_k}$ are vectors in $\mathbb{R}^n$ and $c_1, c_2, ..., c_n$ are scalars, then the vector $\vec{v} = c_1\vec{u_1} + c_2\vec{u_2} + ... + c_k\vec{u_k}$ is called a **linear combination** of the vectors $\vec{u_1},,. . ., \vec{u_k}$.

Example: Given vectors $\vec{u_1} = (3, 4, 6)$ and $\vec{v_2} = (-1, 2, 8)$, then $\vec{v} = 3\vec{u_1} - 2\vec{u_2} = (11, 8, 12)$ is a **linear combination** of $\vec{u_1}$ and $\vec{u_2}$.

Is $\vec{v}$ a linear combination of given vectors?
To determine if a vector $\vec{v}$ is a linear combination of some vectors $\vec{u_1},,. . ., \vec{u_k}$, you must set up a system of linear equations and solve in the normal manner.

Step 1: Set up the system of equations $c_1\vec{u_1} + c_2\vec{u_2} + ... + c_3\vec{u_k} = \vec{v}$
Step 2: Write the system of equations as an augmented matrix (with each vector as a column vector).
Step 3: Reduce the augmented matrix to RREF
Step 4: If the system is consistent, read the solution off the matrix; if the matrix is inconsistent, then $\vec{v}$ cannot be written as a linear combination of the given vectors.

Example: Is the vector $\vec{v} = (2, 6, 9, 1)$ a linear combination of $\vec{u_1} = (1, 2, 1, 3)$ and $\vec{u_2} = (-1, 0, 4, 1)$?
Step 1: Are there a c_1 and a c_2 such that $c_1\vec{u_1} + c_2\vec{u_2} = \vec{v}$?

Step 2: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 0 & 6 \\ 1 & 4 & 9 \\ 3 & 1 & 1 \end{array}\right]$

Step 3: $\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 5 & 7 \\ 0 & 4 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 5 & 7 \\ 0 & 0 & -9 \end{array}\right]$

Step 4: This matrix produces a contradiction, so we can see that the system is inconsistent. Therefore now, $\vec{v}$ cannot be written as a linear combination of $\vec{u_1}$ and $\vec{u_2}$.

Example: Is the vector $\vec{v} = (2, 6, 7)$ a linear combination of $\vec{u_1} = (1, 2, 1)$ and $\vec{u_2} = (-1, 0, 4)$?
Step 1: Are there a c_1 and a c_2 such that $c_1\vec{u_1} + c_2\vec{u_2} = \vec{v}$?

Step 2: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 0 & 6 \\ 1 & 4 & 7 \end{array}\right]$

Step 3: $\underset{R_3 - R_1}{\xrightarrow{R_2 - 2R_1}} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \end{array}\right] \underset{R_3 - 5R_2}{\xrightarrow{R_2(\frac{1}{2})}} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

Step 4: This matrix is consistent, and we can see that $c_1 = 3$ and $c_2 = 1$. So $\vec{v} = 3\vec{u_1} + 1\vec{u_2}$
That is, $(2, 6, 7) = 3(1, 2, 1) + 1(-1, 0, 4)$.

Theorem: $AX = B$ is consistent if and only if B is expressible as a linear combination of the column vectors of A.

Spans

Definition: The **span** of a set of vectors $\vec{u_1}, \vec{u_2}, ..., \vec{u_k}$ in $\mathbb{R}^n$ is the set of all vectors $c_1\vec{u_1} + c_2\vec{u_2} + ... + c_k\vec{u_k}$ that are linear combinations of the given vectors.

Example: Given $\vec{v} = (-1, 1, 20, 16), \vec{u_1} = (1, 3, 6, 4)$, and $\vec{u_2} = 2, 4, -1, 2)$, we can see that $\vec{v}$ is in $span\{u_1, u_2\}$ because it can be written as a linear combination of $\vec{u_1}$ and $\vec{u_2}$ in the following way: $\vec{v} = 3\vec{u_1} - 2\vec{u_2}$.
This can be shown by following the steps above to determine if $\vec{v}$ is a linear combination of $\vec{u_1}$ and $\vec{u_2}$.

Theorem: Because we can always just let all of the weights (constants) be 0, we can see that every span contains the zero vector, $\vec{0}$.

Describing the Span of Vectors - the span of vectors is almost always infinite:

In $\mathbb{R}^2$:

* $span\{(0, 0, 0)\} = (0, 0, 0)$
* $span\{\vec{u}\}$ = the set of all scalar multiples of $\vec{u}$ = a line in $\mathbb{R}^2$
* $span\{\vec{u}, \vec{v}\}$ =the set of all linear combinations of $\vec{u}$ and $\vec{v}$= $\mathbb{R}^2$ if $\vec{u}$ and $\vec{v}$ are not collinear (that is, they are not scalar multiples; they are linearly independent)

In $\mathbb{R}^3$:

* $span\{(0, 0, 0)\} = (0, 0, 0)$
* $span\{\vec{u}\}$ =the set of all scalar multiples of $\vec{u}$ = a line in $\mathbb{R}^3$
* $span\{\vec{u}, \vec{v}\}$ =the set of all linear combinations of $\vec{u}$ and $\vec{v}$ = a plane in $\mathbb{R}^3$ if $\vec{u}$ and $\vec{v}$ are not collinear (that is, they are not scalar multiples; they are linearly independent)
* $span\{\vec{u}, \vec{v}, \vec{w}\}$ =the set of all linear combinations of $\vec{u}, \vec{v}$, and $\vec{w}$ = $\mathbb{R}^3$ if $\vec{u}, \vec{v}$, and $\vec{w}$ are not coplanar (they are linearly independent).

Steps to determine the span of vectors in $\mathbb{R}^n$

Step 1: Set up an augmented matrix with the vectors as column vectors and variables on the right side of the augmentation line.
Step 2: Reduce the matrix to RREF
Step 3: If there is a pivot in every ROW, then the span of the given vectors is $\mathbb{R}^n$
If there is a row of the form $\begin{bmatrix} 0 & 0 & ... & 0 & | & \text{x, y, z} \end{bmatrix}$, then the equation formed from setting the x, y, z equal to 0 will describe the span.

Theorem: Given vectors $\vec{u_1}, \vec{u_2}, ..., \vec{u_k}$ in $\mathbb{R}^n$, if $k = n$ (that is, the number of vectors is the same as the number of elements in the vector) and if the determinant of matrix formed from the column vectors is not 0 (that is, they are linearly independent), then the vectors span $\mathbb{R}^n$.

Example: Describe the span of the vectors $\vec{u_1} = (-1, 4, 3), \vec{u_2} = (3, -2, 6)$, and $\vec{u_3} = (1, 0, 3)$. That is, find an equation relating x, y, and z for general vector $\vec{b} = (x, y, z)$ in $span\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$

Step 1: $\left[\begin{array}{ccc|c} -1 & 3 & 1 & x \\ 4 & -2 & 0 & y \\ 3 & 6 & 3 & z \end{array}\right]$

Step 2: $\xrightarrow[R_3+3R_1]{R_2+4R_1} \left[\begin{array}{ccc|r} -1 & 3 & 1 & x \\ 0 & 10 & 4 & 4x+y \\ 0 & 15 & 6 & 3x+z \end{array}\right] \xrightarrow[(R_2)(\frac{1}{10})]{(R_1)(-1)} \left[\begin{array}{ccr|r} 1 & -3 & -1 & -x \\ 0 & 1 & 2/5 & \frac{4x+y}{10} \\ 0 & 15 & 6 & 3x+z \end{array}\right] \xrightarrow[R_3-15R_2]{R_1+3R_2}$

$\left[\begin{array}{ccr|r} 1 & 0 & 1/5 & \frac{2x+3y}{10} \\ 0 & 1 & 2/5 & \frac{-26x+y}{10} \\ 0 & 0 & 0 & \frac{-6x-3y+2z}{2} \end{array}\right]$

Step 3: There is clearly not a pivot in every row, so these three vectors are coplanar (and thus do not span $\mathbb{R}^3$). We must therefore form an equation from the right side of the row of 0s by setting it equal to 0.

$\frac{-6x-3y+2z}{2} = 0 \to -6x - 3y + 2z = 0$

So $span\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ is the plane $-6x - 3y + 2z = 0$.

Another way this question could be asked:

Example: Find the plane spanned by the vectors $\vec{u} = (1, 3, 3)$ and $\vec{v} = (-1, 2, 0)$.

Step 1: $\left[\begin{array}{cr|c} 1 & -1 & x \\ 3 & 2 & y \\ 3 & 0 & z \end{array}\right]$

Step 2: $\xrightarrow[R_3-3R_1]{R_2-3R_1} \left[\begin{array}{cr|r} 1 & -1 & x \\ 0 & 5 & y-3x \\ 0 & 3 & z-3x \end{array}\right] \xrightarrow{R_2(\frac{1}{5})} \left[\begin{array}{cr|r} 1 & -1 & x \\ 0 & 5 & \frac{y-3x}{5} \\ 0 & 3 & -3x+3y-8z \end{array}\right] \xrightarrow[R_3-3R_2]{R_1+R_2} \left[\begin{array}{cc|r} 1 & 0 & \frac{2x+y}{5} \\ 0 & 1 & \frac{y-3x}{5} \\ 0 & 0 & \frac{-6x-3y+5z}{5} \end{array}\right]$

Step 3: $\frac{-6x-3y+5z}{5} = 0 \to -6x - 3y + 5z = 0$, So the plane spanned by these vectors is $-6x - 3y + 5z = 0$.