Matrix Subspaces

There are two very special subspaces that we see over and over again. These subspaces are the **Null Space** and the **Column Space** of a matrix, A.

Definition: The **null space** of an $m \times n$ matrix, A, written Nul(A), is the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

Definition: The **column space** of a matrix A, written col(A) is the subspace spanned by the column vectors of A. That is, if A has column vectors $\vec{a_1}, \vec{a_2}, ..., \vec{a_k}$, then the column space of A is the set of all linear combinations $c_1\vec{a_1} + c_2\vec{a_2} + ... + c_k\vec{a_k}$.

Theorem: The linear system $AX = B$ is consistent if and only if B is in the column space of A. That is, if and only if B can be written as a linear combination of the columns of A.

Example: Consider $A = \begin{bmatrix} 3 & 1 & 2 & 4 \\ -3 & 4 & 2 & 1 \\ 6 & 2 & 2 & 7 \end{bmatrix}$

The matrix A reduces to $R = \begin{bmatrix} 1 & 0 & 0 & 4/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$.

From this, we can see that $\text{Nul}(A) = \{(-4/5, -3/5, -1/2, 1)\}$.

Also, $\text{Col}(A) = span\{(3,-3,6),(1,4,2),(2,2,2),(4,1,7)\} = span\{(3,-3,6),(1,4,2),(2,2,2)\}$

Theorem: The relationship between the column of the original matrix is the same as the relationship between the column vectors of the row reduced matrix.
Basically, this means that, to determine the relationship between any of the columns, you can take the columns from the row-reduced form, create a new matrix, and finish the row-reduction.

Definition: Columns that represent basis vectors for Col(A) are called **basic columns**. Columns that do not are called **non-basic columns**.

Dimension Theorem

Definition: The **rank** of a matrix A = the dimension of col(A) = number of pivots
Definition: The **nullity** of a matrix A = the dimension of Nul(A) = number of non-basic columns

Theorem: Let A be a matrix with n columns. Then $\text{rank} + \text{nullity} = n$.

Example: Let $A = \begin{bmatrix} 7 & 1 & 4 & 1 & 8 \\ -2 & 0 & -2 & 0 & -2 \\ 1 & 5 & -14 & 1 & 10 \\ 3 & 4 & -9 & 1 & 1 \end{bmatrix}$ reduce to $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a) Find a basis for Col(A).
Looking at R, we can see that the columns with pivots (the basic columns) are $\vec{c_1}, \vec{c_2}$, and $\vec{c_4}$; the 3 vectors from the original matrix are therefore linearly independent and will form a basis for Col(A).

basis for Col(A)=$\{(7, -2, 1, 3), (1, 0, 5, 4), (1, 0, 1, 1)\}$

(Note that this is NOT the only basis for Col(A). Looking at R again, let us consider the first, second and fifth columns. Do these also form linearly independent vectors? We can determine this by continuing the row reduction from R.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can see that they are linearly independent, so another possible basis for Col(A) is $\{(7, -2, 1, 3), (1, 0, 5, 4), (8, -2, 10, 10)\}$.

b) Find rank(A).
Every basis for Col(A) contains 3 vectors, so $\text{rank}(A) = 3$.

c) Find a basis for Nul(A).
Let us use R to come up with the parametric solutions to $AX = 0$, representing the null space:

$$\begin{cases} x_1 = -t - s \\ x_2 = 3t - 2s \\ x_3 = t \\ x_4 = s \\ x_5 = s \end{cases}$$

, where x_3 and x_5 are free variables. Reading from this, we can see that a basis for Nul(A) is $\{(-1, 3, 1, 0, 0), (-1, -2, 0, 1, 1)\}$.

d) Find nullity(A).
Every basis for Nul(A) contains 2 vectors, so $\text{nullity}(A) = 2$.

e) Show that the dimension theorem holds?
rank+nullity=3+2=5. The original matrix has 5 columns, so we can see that the dimension theorem holds.

f) Can $\vec{c_2}$ be written as a linear combination of $\vec{c_1}$ and $\vec{c_3}$?
Set up the necessary matrix and continue the row reduction from R to find out!

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Yes! $(1/3)c_1 - (1/3)c_3 = c_2 : (1/3)(7, -2, 1, 3) - (1/3)(4, -2, -14, -9) = (1, 0, 5, 4)$

Theorem: Let A be an $m \times n$ matrix. Then $\text{rank}(A^T) = \text{rank}(A)$ and $\text{nullity}(A^T) = m - \text{rank}(A)$

Consult the following table of examples to understand the relationship between the size of a matrix, its rank, and its nullity:

Size of A	4×4	4×4	4×4	4×4	7×6	8×3
Rank(A)	4	3	2	1	2	2
Dim of Col(A)	4	3	2	1	2	2
Dim of Nul(A)	0	1	2	3	4	1
Dim of Nul(A^T)	0	1	2	3	5	6

Theorem: Let A be an $n \times n$ matrix. Then if $\det(A) \neq 0$, $\text{rank}(A) = n$. If $\det(A) = 0$, then $\text{rank}(A) \neq n$.

Example: Let A, B, and C be 3×3 matrices. Let $\det(A) = 8$, $\det(B) = -2$, and let C be singular (not invertible).

a) Find $\det(AC+BC)$: $\det(AC+BC) = \det((A+B)C) = \det(A+B)\det(C) = det(A+B)(0) = 0$

b) Find Col(A): $\det(A) \neq 0$, so $\text{rank}(A) = 3$. Therefore Col(A)=$\mathbb{R}^3$, because the columns are linearly independent.

c) Find Nul(A): nullity(A)=0 because rank(A)=3, so $Nul(A) = \{(0,0,0)\}$

Example: Let A be a 9×4 matrix.

a) Is it possible for $AX = 0$ to have a unique solution?
Yes, because there are enough rows for every column to have a pivot.

b) Is it possible for $A^T X = 0$ to have a unique solution?
No. A^T is a 4×9 matrix. There would need to be a pivot in every column for it to have a unique solution, but there aren't enough rows for every column to have a pivot.

c) What is the minimum possible value for the nullity of A^T? We know that rank(A^T) =rank(A). We also know that nullity(A^T)=9-rank(A). Because it is possible for $AX = 0$ to have a unique solution, the maximum possible value for rank(A) is 4, representing full rank. Therefore the minimum possible value for nullity(A^T) is 9-4=5.

d) If rank(A)=2, how many parameters are there in the solution to $A^T X = \vec{0}$?
We know rank(A^T) =rank(A) = 2. We also know that A^T is a 4×9 matrix. The rank being 2 tells us that there are 2 basic columns. Since there are 9 columns in total, there must therefore be 7 non-basic columns. And therefore there will be 7 paramaters in the solution to $A^T X = 0$.

e) If rank(A) = 0, what is Nul(A) and what is Col(A)?
If the rank of A is 0, then Col(A) is just the 0-vector in $\mathbb{R}^9$.
This means Nul(A) has dimension 4 and is in $\mathbb{R}^4$. Four linearly independent vectors span the space, so Nul(A) = $\mathbb{R}^4$.

Every subspace is the solution space to some homogeneous system of equations.

Theorem: The general solution to $AX = B$ is found by adding a particular solution to $AX = B$ to the general solution of $AX = 0$.

Example: Consider the matrix$A = \begin{bmatrix} 1 & -2 & 7 \\ 3 & 6 & -3 \\ 5 & 1 & 13 \end{bmatrix}$ which reduces to $R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. Then I can find Nul(A) by solving $AX = 0$. $\begin{cases} x = -3t \\ y = 2t \\ z = t \end{cases}$, so $\text{Nul}(A) = \text{span}\{(-3, 2, 1)\}$.

Now, let $x_0 = (4, -1, 3)$ be a particular solution to $AX = B$ (that is, it is one possible vector that satisfies $AX + B$). Then the general solution to $AX = B$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

EQUIVALENT STATEMENTS: The following assertions are equivalent for an $n \times n$ matrix A.

a) A is invertible
b) $AX = 0$ has only the trivial solution
c) A reduces to I
d) $AX = B$ is consistent for every $n \times 1$ matrix B
e) $AX = B$ has a unique solution for every $n \times 1$ matrix B.
f) $\det(A) \neq 0$
g) The column vectors of A are linearly independent
h) The column vectors of A span $\mathbb{R}^n$
i) The column vectors of A form a basis for $\mathbb{R}^n$
j) A has rank n (and $\text{Col}(A) = \mathbb{R}^n$)
k) A has nullity 0 (and $\text{Nul}(A) = \vec{0} \in \mathbb{R}^n$)