**Simplex Method Notes**

**Introduction to the Simplex Algorithm**: In three variables, it is difficult to sketch the feasibility region; in four or more variables, it is impossible. The **simplex algorithm** gives a way to solve linear programs without sketching. We introduce it on an easy example.

Ex : Solve the linear program:

$$\begin{array}{lrcrcrcr} \text{Maximize} & z & = & 3x_1 & + & 2x_2 & & \\ \text{subject to} & & & x_1 & + & x_2 & \leq & 3 \\ & & & 2x_1 & + & x_2 & \leq & 4 \\ & & & x_1 & \geq 0, & x_2 & \geq 0 & \end{array}$$

First express the objective function homogeneously: $z - 3x_1 - 2x_2 = 0$.

Introduce slack variables to modify constraints:

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & s_1 & & & = & 3 \\ 2x_1 & + & x_2 & + & & & s_2 & = & 4 \\ & x_1 \geq 0, & & x_2 \geq 0, & & s_1 \geq 0, & & s_2 \geq 0 & \end{array}$$

Set-up initial simplex table:

$$\left[\begin{array}{c|cc|cc|c} 1 & -3 & -2 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 2 & 1 & 0 & 1 & 4 \end{array}\right]$$

Each simplex table determines a **Basic Feasible Solution** (b.f.s) as follows : A variable whose column has exactly one non-zero entry is **basic**, otherwise it is **non-basic**. Set non-basic variables to zero and solve for basic variables to get b.f.s.

In the table above, $z$, $s_1$ and $s_2$ are basic. $x_1$ and $x_2$ are non-basic. Thus the associated b.f.s. has $z = 0$ at $(x_1, x_2, s_1, s_2) = (0, 0, 3, 4)$.

Pivot in the 2nd column, 3rd row:

$$\left[\begin{array}{c|cc|cc|c} 2 & 0 & -1 & 0 & 3 & 12 \\ \hline 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 1 & 4 \end{array}\right]$$

with b.f.s. $z = 6$ at $(2, 0, 1, 0)$.

Pivot in the 3rd column, 2nd row:

$$\left[\begin{array}{c|cc|cc|c} 2 & 0 & 0 & 2 & 2 & 14 \\ \hline 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 2 & 0 & -2 & 2 & 2 \end{array}\right]$$

with b.f.s. $z = 7$ at $(1, 2, 0, 0)$, the maximum.

Question 1: When to stop pivoting?
The maximum is reached when there are no negatives in the top row.

Question 2: How to choose pivot?

- To choose pivot column, take any column with a negative in the top row. (Entry furthest from 0 often leads to answer fastest.)
- To choose pivot row, divide each entry in last column by corresponding entry in pivot column. Smallest positive ratio determines pivot row.

$$\begin{array}{llrcrcrl} \text{Ex:} & \text{Maximize} \quad z = & 18x_1 & + & 20x_2 & + & 32x_3 & \\ & \text{subject to} & x_1 & + & 2x_2 & + & 2x_3 & \leq 22 \\ & & 3x_1 & + & 2x_2 & + & 4x_3 & \leq 40 \\ & & 3x_1 & + & x_2 & + & 2x_3 & \leq 14 \\ & & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 \end{array}$$

$$\begin{array}{llrcrcrl} \text{Ex:} & \text{Maximize} \quad z = & -15x_1 & - & 14x_2 & + & 24x_3 & \\ & \text{subject to} & 4x_1 & - & 2x_2 & + & 2x_3 & \leq 2 \\ & & 16x_1 & - & 2x_2 & + & 4x_3 & \leq 10 \\ & & 15x_1 & - & 5x_2 & + & 6x_3 & \leq 15 \\ & & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 \end{array}$$

**More Simplex Method Notes**

**The Simplex Algorithm** (continued):

**Exercise:**

$$\begin{array}{llll} \text{Maximize } z = & -6x_1 & -\ 2x_2 & +\ \ x_3 \\ \text{subject to} & 6x_1 & -\ 5x_2 & +\ 2x_3 \leq 5 \\ & 2x_1 & -\ 3x_2 & +\ \ x_3 \leq 2 \\ & 6x_1 & -\ 4x_2 & +\ 2x_3 \leq 10 \\ & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0 \end{array}$$

**Ex**:Leather Products Ltd. has cutting machines, sewing machines, and a supply of leather with which they make shoes, purses, and coats. Suppose that the articles use the machines and leather as given in the table below:

| | Shoes | Purses | Coats |
|---|---|---|---|
| Cutting machines (10 min) | 1 | 2 | 5 |
| Sewing machines (10 min) | 4 | 1 | 5 |
| Leather (Square ft) | 2 | 4 | 10 |

Further, let there be a profit of \$22 on shoes, \$15 on purses, and \$50 on each coat. Suppose also that there are 3000 minutes available on the cutting machines, 4050 minutes on the sewing machines, and 900 square feet of leather. Set up and solve the linear program to obtain the maximum profit from the available resources. When the maximum profit of reached, how much of each resource goes unused?

**Ex:** Pauls Pipe and Tabacco Shop blends Virginia, Turkish, and Mexican Tabaccos into three mixtures: Wild, Heather, and Silk. A packet of Wild has 6 ounces of Virginia, 4 ounces of Turkish, and 14 ounces of Mexican tabacco; a packet of Heather has 3 ounces of Virginia, 1 ounce of Turkish, and 2 ounces of Mexican tabacco; and one packet of Silk has 4 ounces of Virginia, 2 ounces of turkish, and 8 ounces of Mexican.

If Wild sells for \$8, Heather for \$10, and Silk for \$16 per packet, and if there are 32 ounces of Viriginia, 10 ounces of Turkish, and 72 ounces of Mexican on hand, set up and solve the linear program that maximizes revenue. When revenue is maximized, how much of each tabacco is **left over?**

**Simplex Algorithm: Considerations**

While applying the maximum simplex algorithm, if you encounter **any** simplex table with a column which has a negative on the top row, (so it is a suitable pivot column), but no positive entry below (so no possible pivot), then the region is **unbounded** and the maximum simply does not exist.

Ex: Sketch the feasibility region in the following problem. Then try the simplex algorithm to see what happens.

$$\begin{array}{llll} \text{Maximize } z = & 2x_1 & + & x_2 \\ \text{subject to} & -2x_1 & + & x_2 \leq 1 \\ & x_1 & - & x_2 \leq 1 \\ & x_1 \geq 0, & x_2 \geq 0 & \end{array}$$

In such cases, we instead can ask for a feasible solution with $z \geq k$ for some number $k$. The logic is that if we can't maximize $z$, at least we can figure out how to make sure $z$ is big enough for our purposes.

To do this, we work with the simplex table where you noticed the unboundedness. Set all the nonbasic variables **except** for the variable of the affected column (the one we can't find a pivot for) to zero and solve for everything else. This time, you will end up with parameters in your solution, but an appropriate choice will ensure $z$ is sufficiently large.

Ex: Show that the linear program has no maximum. Then find a feasible solution with $z \geq 30000$.

$$\begin{array}{lrrrcl} \text{Maximize} & -6x_1 & +\, 4x_2 & -\, x_3 & = & z \\ \text{subject to} & -2x_1 & +\, x_2 & -\, x_3 & \leq & 3 \\ & x_1 & -\, 2x_2 & +\, 2x_3 & \leq & 3 \\ & -x_1 & +\, 2x_2 & -\, 4x_3 & \leq & 8 \\ & \multicolumn{5}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} \end{array}$$

Ex: Show that the linear program has no maximum. Then find a feasible solution with $z \geq 1000$.

$$\begin{array}{lrrrcl} \text{Maximize} & 3x_1 & -\, x_2 & -\, 4x_3 & = & z \\ \text{subject to} & 6x_1 & -\, 19x_2 & +\, 19x_3 & \leq & 13 \\ & 3x_1 & -\, 9x_2 & +\, 9x_3 & \leq & 6 \\ & -3x_1 & +\, 10x_2 & -\, 11x_3 & \leq & 7 \\ & \multicolumn{5}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} \end{array}$$

**Degeneracy**:

If, when choosing a pivot row there is a tie in the ratios, the system is considered **degenerate**. Either pivot may be chosen and the next simplex table will contain a zero in the right column.

Ex: Solve the linear program.

$$\begin{array}{lrrrl} \text{Maximize } z = & 8x_1 & -\, 7x_2 & -\, 16x_3 & \\ \text{subject to} & 2x_1 & -\, 2x_2 & -\, 4x_3 & \leq 4 \\ & 4x_1 & -\, 3x_2 & -\, 10x_3 & \leq 8 \\ & -2x_1 & +\, 2x_2 & +\, 6x_3 & \leq 2 \\ & \multicolumn{4}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} \end{array}$$

While the algorithm might terminate and give an answer, it is possible–though rare–for the algorithm to go into an infinite loop.

**An example of cycling**:

If in the following problem we choose the first constraint over the second when there is a tie in the ratios, the algorithm will never terminate.

$$\begin{array}{lrrrrl} \text{Maximize} & 2x_1 & +\, 3x_2 & -\, x_3 & -\, 12x_4 & = z \\ \text{subject to} & -2x_1 & -\, 9x_2 & +\, x_3 & +\, 9x_4 & \leq 0 \\ & \frac{x_1}{3} & +\, x_2 & -\, \frac{x_3}{3} & -\, 2x_4 & \leq 0 \\ & \multicolumn{5}{c}{\text{Each } \ x_i \geq 0} \end{array}$$

**The Generalized Simplex Algorithm**:

So far we have discussed how to use the simplex algorithm to **maximize** linear functions subject to constraints of the form: *linear* $\leq$ *positive number.* Among the many variations to the algorithm, we will discuss only how to **minimize**.

To do this, one need only make one small but important observation. A function attains its maximum EXACTLY when its opposite attains its minimim, i.e. to minimize $z$, we simply need to maximize $-z$.

Thus we set up our initial simplex table as before (except) we use a new variable $\bar{z} = -z$ to take the place $z$. Make sure the coefficient in front of $\bar{z}$ is positive and set everything else up accordingly. Now proceed with the simplex algorithm as before. When we get an answer for the maximum of $\bar{z}$, we only need to remember that $z$ has its minimum at the opposite of that value.

Ex: Solve the linear program.

$$\begin{array}{lrrrl} \text{Minimize } z = & 4x_1 & -\, 10x_2 & +\, 6x_3 & \\ \text{subject to} & -10x_1 & +\, 4x_2 & +\, 8x_3 & \leq 20 \\ & -2x_1 & +\, 2x_2 & +\, 4x_3 & \leq 6 \\ & 4x_1 & -\, 2x_2 & +\, 2x_3 & \leq 4 \\ & \multicolumn{4}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} \end{array}$$

Ex: Solve the linear program

$$\begin{array}{lrrrl} \text{Minimize } z = & -8x_1 & +\, x_2 & +\, 2x_3 & \\ \text{subject to} & x_1 & +\, 3x_2 & -\, x_3 & \leq 3 \\ & 3x_1 & +\, 3x_2 & & \leq 10 \\ & 5x_1 & +\, 10x_2 & -\, 2x_3 & \leq 20 \\ & \multicolumn{4}{c}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} \end{array}$$