Subspaces

Definition: A nonempty subset S in $\mathbb{R}^n$ is called a **subspace** if and only if the following properties are satisfied:

1) If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S. That is, S is **closed under vector addition**. (VA)
2) If $\vec{u}$ is in S and if c is any scalar, then $c\vec{u}$ is also in S. That is, S is **closed under scalar multiplication**. (SM)

In other words, if S is a subspace, then any linear combination of vectors in S will remain in S. Also, because the constants in the linear combination can always be 0, **EVERY** subspace must contain $\vec{0}$.

Theorem: A nonempty subset S of $\mathbb{R}^n$ is a subspace if and only if it is the span of some finite set of vectors $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$ in $\mathbb{R}^n$. (That is, if it's a subspace, it's a span, and if it's a span, it's a subspace. It goes both ways!)

Recognizing Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

Subspaces of $\mathbb{R}^2 : \{(x,y)\}$	**Subspaces of** $\mathbb{R}^3 : \{(x,y,z)\}$
$\{(0,0)\}$ ONLY if both x and y are 0	$\{(0,0,0)\}$ ONLY if x, y, and z are all 0
Line through $(0,0)$ Slope-Intercept Form: $y = mx$ General Form: $ax + by = 0$ Parametric Form: $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$ Vector Form: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \end{bmatrix}$	**Line through** $(0,0,0)$ Parametric Form: $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$ Vector Form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} + s \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ **Plane Through** $(0,0,0)$ General Form: $ax + by + cz = 0$ Parametric Form: $\begin{cases} x = x_0 + ta_1 + sa_2 \\ y = y_0 + tb_1 + sb_2 \\ z = z_0 + tc_1 + sc_2 \end{cases}$ Vector Form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + s \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$
$\mathbb{R}^2$ - every single vector of the form (x,y)	$\mathbb{R}^3$ - every single vector of the form (x,y,z)

Methods for Determining if something is a Subspace

Method 1:

1. Check to see if $\vec{0} \in S$. If it is not, S is not a subspace, and you are done.

2. Write two generic vectors in the space, $\vec{v_1} = (x_1, y_1, z_1)$ and $\vec{v_2} = (x_2, y_2, z_2)$.
3. Show that Vector Addition (VA) holds - add the two vectors together and show that they still maintain the properties necessary to be in S.

4. Show that Scalar Multiplication holds (SM) - multiply one of the generic vectors by a scalar k and show that it still maintains the properties necessary to be in S.

5. If either VA or SM fails, S is not a subspace, and you must provide a counter example.

Method 2:

1. Check to see if $\vec{0} \in S$. If it is not, S is not a subspace, and you are done.

2. Check to see if S is the span of some vectors in $\mathbb{R}^n$ - using the above chart will be helpful for this.
3. If yes to both, S is a subspace. If no to even one of the questions, S is not a subspace, and you must provide a counterexample.

Note: Sometimes it is not obvious whether or not S can be written as a span of vectors. In this case, you must use Method 1.

Example: Show that each of the following satisfies the closure properties and is therefore a subspace of $\mathbb{R}^n$. Also, write them as the span of a finite set of vectors.

a) $S = \{(x, y, z) \in \mathbb{R}^3 | x = 4t, y = -2t, z = t\}$

Method 1:

1. If $t = 0$ then $(x, y, z) = (0, 0, 0)$, so $\vec{0} \in S$.

2. $\vec{v_1} = (x_1, y_1, z_1)$ where $x_1 = 4t_1, y_1 = -2t_1, z_1 = t_1$ and $\vec{v_2} = (x_2, y_2, z_2)$ where $x_2 = 4t_2, y_2 = -2t_2, z_2 = t_2$

3. VA: $\vec{v_1} + \vec{v_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ We need $x_1 + x_2 = 4t, y_1 + y_2 = -2t$, and $z_1 + z_2 = t$
$x_1 + x_2 = 4t_1 + 4t_2 = 4(t_1 + t_2), y_1 + y_2 = -2t_1 - 2t_2 = -2(t_1 + t_2)$, and $z_1 + z_2 = t_1 + t_2$
We can let $t = t_1 + t_2$, and we see that vector addition holds.

4. SM: $k\vec{v} = (kx_1, kx_2, kx_3)$ We need $kx_1 = 4t, ky_1 = -2y$, and $kz_1 = t$

   $kx_1 = k(2t_1) = 2(kt_1), ky_1 = k(-2t_1) = -2(kt_1)$, and $kz_1 = kt_1$
   We can let $t = kt_1$ and scalar multiplication holds.

So S is a subspace. Now we want to write it as a span of vectors. We know that $\begin{cases} x &= 4t \\ y &= -2t \\ z = t \end{cases}$ is the parametric equation for a line going through $(0, 0, 0)$ with parallel vector $(4, -2, 1)$. So this subspace equals $span\{(4, -2, 1)\}$.

b) $S = \{(x, y, z) \in \mathbb{R}^3 | 3x - 4y + 5z = 0\}$

Method 1:

1. $(0, 0, 0)$ satisfies $3x - 4y + 5z = 0$, so $\vec{0} \in S$

2. $\vec{v_1} = (x_1, y_1, z_1)$ where $3x_1 - 4y_1 + 5z_1 = 0$ and $\vec{v_2} = (x_2, y_2, z_2)$ where $3x_2 - 4y_2 + 5z_2 = 0$

3. $\vec{v_1} + \vec{v_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ We need $3(x_1 + x_2) - 4(y_1 + y_2) + 5(z_1 + z_2) = 0$

   $= 3x_1 + 3x_2 - 4y_1 - 4y_2 + 5z_1 + 5z_2 = 3x_1 - 4y_1 + 5z_1 + 3x_2 - 4y_2 + 5z_2 = 0 + 0 = 0$
   We can see that vector addition holds.

4. $k\vec{v_1} = (kx_1, ky_1, kz_1)$ $3kx_1 - 4ky_1 + k5z_1 = k(3x_1 - 4y_1 + 5z_1) = 0$ We can see that scalar multiplication holds.

So S is a subspace. We must now find the span, and we can do this by setting up a matrix and finding the solution space.

$\begin{bmatrix} 3 & -4 & 5|0 \end{bmatrix}$ So y and z are free variables, and we can see that $\begin{cases} x & = 4/3t - 5/3s \\ y & = t \\ z & = s \end{cases}$ so that

$S = span\{(4/3, 1, 0), (-5/3, 0, 1)\}$.

Note: The zero subspace $\{\vec{0}\}$ and $\mathbb{R}^n$ itself are always subspaces of $\mathbb{R}^n$.