SEQUENCES

Convergence

A sequence $\{a_n\}$ converges if $\lim a_n$ exists and is finite.

Squeeze theorem

If $b_n \leq a_n \leq c_n$ for all values of n, and $\lim b_n = \lim c_n = L$, then it implies that $\lim a_n = L$.

Other Useful facts

a_n converges to zero if and only if $|a_n|$ also converges to zero.

When n is large, $\ln(n) < n^p < a^n < n! < n^n$

SERIES

Partial sums

$$s_N = \sum_{n=1}^{N} a_n$$

Convergence

A series is *convergent* when the limit of partial sums exists,

$$\sum a_n = \lim_{N \to \infty} s_N$$

otherwise it is *divergent*.

A series is *absolutely convergent* when $\sum |a_n|$ is convergent.

A series is *conditionally convergent* when $\sum |a_n|$ is divergent but $\sum a_n$ is convergent.

Geometric series

$\sum ar^{n-1}$ converges when $|r| < 1$, otherwise diverges.
When convergent, the sum is equal to $\dfrac{a}{1-r}$.

p-series

$\sum \dfrac{1}{n^p}$ converges when $p > 1$, otherwise diverges.

Divergence Test

If $\lim a_n \neq 0$, then the series $\sum a_n$ is divergent.

Integral Test

If $a_n = f(n)$ when $f(x)$ is a positive, continuous, eventually decreasing function, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)dx \text{ converges}$$

Comparison Test

Suppose a_n and b_n are two positive sequences, with $a_n \leq b_n$ for all $n > N$ for some number N.

If $\sum b_n$ is convergent, then so is $\sum a_n$.

If $\sum a_n$ is divergent, then so is $\sum b_n$.

Limit Comparison Test

Suppose that a_n and b_n are two positive sequences, and $\lim \frac{a_n}{b_n} = c$.

- If $c > 0$ is a finite number, then

$$\sum a_n \text{ converges} \iff \sum b_n \text{ converges.}$$

- If $c = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $c = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ratio Test

Suppose that $\lim \left| \dfrac{a_{n+1}}{a_n} \right| = L$.

- If $L < 1$, then $\sum a_n$ is absolutely convergent.
- If $L > 1$, then $\sum a_n$ is divergent.
- If $L = 1$, then the test is inconclusive.

Root Test

Suppose that $\lim |a_n|^{1/n} = L$.

- If $L < 1$, then $\sum a_n$ is absolutely convergent.
- If $L > 1$, then $\sum a_n$ is divergent.
- If $L = 1$, then the test is inconclusive.

Alternating Series Test

For series of the form $\sum(-1)^n b_n$, where b_n is a positive and eventually decreasing sequence, then

$$\sum (-1)^n b_n \text{ converges} \iff \lim b_n = 0$$

POWER SERIES

Definitions

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{OR} \quad \sum_{n=0}^{\infty} c_n (x-a)^n$$

Radius of convergence: The radius is defined as the number R such that the power series converges if $|x-a| < R$, and diverges if $|x-a| > R$.

Interval of convergence: I = interval of values of x for which the power series is convergent. Note that the length of the interval is twice the radius of convergence..

MacLaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$