Cal II: Worksheet 4 (rational functions)

1. $\displaystyle\int \frac{dx}{2x^3+x^2-x}$ **2.** $\displaystyle\int \frac{3x^3-5x^2-11x+9}{x^2-2x-3}\,dx$ **3.** $\displaystyle\int \frac{x^2+12x-5}{(x+1)^2(x-7)}\,dx$

4. $\displaystyle\int \frac{8x^2-3x-4}{(4x-1)(x^2+1)}\,dx$ **5.** $\displaystyle\int \frac{4x^3+2x^2+1}{4x^3-x}\,dx$ **6.** $\displaystyle\int \frac{3x-2}{x^3+x^2-x-1}\,dx$

7. $\displaystyle\int \frac{6x^2-x-1}{3x-1}\,dx$ **8.** $\displaystyle\int \frac{3x+5}{x^2+4x+13}\,dx$

Answers and comments

1. $\frac{2}{3}\ln|2x-1|+\frac{1}{3}\ln|x+1|-\ln|x|+C$. The required partial fractions decomposition has the form

$$\frac{1}{x(2x-1)(x+1)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+1}$$

so we have

$$1=A(2x-1)(x+1)+Bx(x+1)+Cx(2x-1)$$

Putting $x=0,\ \frac{1}{2},\ -1$ gives $A=-1$, $B=\frac{4}{3}$ and $C=\frac{1}{3}$.

2. $\frac{3}{2}x^2+x+3\ln|x-3|-3\ln|x+1|+C$. Long division shows that the integrand equals

$$3x+1+\frac{12}{x^2-2x-3}$$

The last term has a partial fractions decomposition of the form

$$\frac{12}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}$$

so we have

$$12=A(x+1)+B(x-3)$$

Putting $x=3,\ -1$ gives $A=3$ and $B=-3$.

3. $2\ln|x-7|-\ln|x+1|-\dfrac{2}{x+1}+C$. The required partial fractions decomposition has the form

$$\frac{x^2+12x-5}{(x+1)^2(x-7)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-7}$$

so we have

$$x^2+12x-5=A(x+1)(x-7)+B(x-7)+C(x+1)^2$$

Putting $x=-1,\ 7$ gives $B=2$ and $C=2$. Equating the coefficients of x^2 on both sides gives

$$1=A+C=A+2 \implies A=-1$$

4. $\frac{3}{2}\ln(x^2+1)-\ln|4x-1|+C$. The required partial fractions decomposition has the form

$$\frac{8x^2-3x-4}{(4x-1)(x^2+1)}=\frac{A}{4x-1}+\frac{Bx+C}{x^2+1}$$

so we have

$$8x^2-3x-4=A(x^2+1)+(Bx+C)(4x-1)$$

Putting $x=\frac{1}{4}$ gives

$$\tfrac{1}{2}-\tfrac{3}{4}-4=\tfrac{17}{16}A \implies A=-4$$

Equating the coefficients of x^2 and the constant terms on both sides gives

$$\begin{aligned}8&=A+4B=-4+4B \implies B=3\\ -4&=A-C=-4-C \implies C=0\end{aligned}$$

5. $x-\ln|x|+\ln|2x-1|+\frac{1}{2}\ln|2x+1|+C$. Long division shows that the integrand equals

$$1+\frac{2x^2+x+1}{4x^3-x}$$

The last term has a partial fractions decomposition of the form

$$\frac{2x^2+x+1}{x(2x-1)(2x+1)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{2x+1}$$

so we have

$$2x^2+x+1=A(2x-1)(2x+1)+Bx(2x+1)+Cx(2x-1)$$

Putting $x=0,\ \frac{1}{2},\ -\frac{1}{2}$ gives $A=-1$, $B=2$ and $C=1$.

6. $\frac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right|-\dfrac{5}{2(x+1)}+C$. The denominator factors by grouping:

$$\begin{aligned}x^3+x^2-x-1&=x^2(x+1)-(x+1)\\&=(x^2-1)(x+1)=(x+1)^2(x-1)\end{aligned}$$

The required partial fractions decomposition then has the form

$$\frac{3x-2}{(x+1)^2(x-1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}$$

so we have

$$3x-2=A(x+1)(x-1)+B(x-1)+C(x+1)^2$$

Putting $x=-1,\ 1$ gives $B=\frac{5}{2}$ and $C=\frac{1}{4}$. Equating the coefficients of x^2 on both sides gives

$$0=A+C=A+\tfrac{1}{4} \implies A=-\tfrac{1}{4}$$

7. $x^2+\frac{1}{3}x-\frac{2}{9}\ln|3x-1|+C$. Long division shows that the integrand equals

$$2x+\tfrac{1}{3}-\frac{\frac{2}{3}}{3x-1}$$

8. $\frac{3}{2}\ln(x^2+4x+13)-\frac{1}{3}\arctan\big(\frac{1}{3}(x+2)\big)+C$. Since

$$x^2+4x+13=(x+2)^2+9$$

let $u=x+2$, so $x=u-2$ and $dx=du$. The given integral then equals

$$\int\frac{3u-1}{u^2+9}\,du=3\int\frac{u\,du}{u^2+9}-\int\frac{du}{u^2+9}$$