Cal II: Worksheet 5

1. $\int \frac{\ln x}{\sqrt{x}}\,dx$ **2.** $\int_{1/\sqrt{2}}^{1} \arcsin x\,dx$ **3.** $\int \sin^2 x\cos^2 x\,dx$ **4.** $\int x^2 e^{3x}\,dx$

5. $\int \frac{4x-3}{x^2+16}\,dx$ **6.** $\int \frac{\cos^3 x}{\sin^4 x}\,dx$ **7.** $\int \frac{(\ln x)^2}{x}\,dx$ **8.** $\int (\ln x)^2\,dx$

9. $\int \frac{dx}{x^4\sqrt{x^2-1}}$ **10.** $\int \arctan x\,dx$ **11.** $\int \frac{\sin^5 x\,dx}{\cos^4 x}$ **12.** $\int \frac{4x^3-6x^2-5}{2x^2-5x-3}\,dx$

13. $\int_1^2 \ln x\,dx$ **14.** $\int \frac{e^{2x}}{1+e^{4x}}\,dx$ **15.** $\int_0^1 \frac{3x-1}{\sqrt{4-x^2}}\,dx$ **16.** $\int \sin^2(x/4)\cos^2(x/4)\,dx$

17. $\int \frac{x^3\,dx}{\sqrt{25-x^2}}$ **18.** $\int x\,\text{arcsec}\,x\,dx$ **19.** $\int \sin^4 x\,dx$ **20.** $\int \frac{\sqrt{x^2+1}}{x^4}\,dx$

21. $\int e^{2x}\cos x\,dx$ **22.** $\int \arccos x\,dx$ **23.** $\int \sec^4 x\tan^3 x\,dx$ **24.** $\int \sqrt{\sec x}\tan x\,dx$

25. $\int e^{\sqrt{x}}\,dx$ **26.** $\int \frac{dx}{x^4-1}$ **27.** $\int \frac{dx}{(1-x^2)^{5/2}}$ **28.** $\int \sec^4(x/2)\tan^2(x/2)\,dx$

29. $\int_0^1 x\arctan x\,dx$ **30.** $\int_{\sqrt[4]{2}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^4-1}}$ **31.** $\int \frac{\sin^2 x}{\cos x}\,dx$ **32.** $\int \text{arcsec}\,x\,dx$

33. $\int e^x \sin 2x\,dx$ **34.** $\int \sin(\ln x)\,dx$ **35.** $\int \sec^2\sqrt{x}\,dx$ **36.** $\int \frac{dx}{(x^2+1)^2}$

Answers and comments

1. $2(\ln x-2)\sqrt{x}+C$. Integrate by parts with $u=\ln x$ and $dv=x^{-1/2}\,dx$.

2. $\left[x\arcsin x+\sqrt{1-x^2}\right]_{1/\sqrt{2}}^{1}=\frac{1}{2}\pi-(\frac{1}{4}\pi+1)/\sqrt{2}$. Integrate by parts with $u=\arcsin x$ and $dv=dx$.

3. $\frac{1}{8}(x-\frac{1}{4}\sin 4x)+C$. Using half-angle identities, the integrand equals

$$\tfrac{1}{4}(1-\cos^2 2x)=\tfrac{1}{4}\big(1-\tfrac{1}{2}(1+\cos 4x)\big)=\tfrac{1}{8}(1-\cos 4x)$$

4. $\frac{1}{27}(9x^2-6x+2)e^{3x}+C$. Integrate by parts twice, with $dv=e^{3x}\,dx$ both times.

5. $2\ln(x^2+16)-\frac{3}{4}\arctan(x/4)+C$. The integral equals

$$2\int\frac{2x\,dx}{x^2+16}-3\int\frac{dx}{x^2+16}$$

(For the first integral, let $u=x^2+16$.)

6. $-\frac{1}{3}\csc^3 x+\csc x+C$. The integral equals

$$\int\frac{(1-\sin^2 x)}{\sin^4 x}\cos x\,dx=\int\frac{1-u^2}{u^4}\,du=\int(u^{-4}-u^{-2})\,du$$

where $u=\sin x$.

7. $\frac{1}{3}\ln^3 x+C$. Let $u=\ln x$.

8. $x(\ln^2 x-2\ln x+2)+C$. Integrate by parts twice, with $dv=dx$ both times. Compare with number 7.

9. $\frac{\sqrt{x^2-1}}{x}-\frac{(x^2-1)^{3/2}}{3x^3}+C$. Trig substitution: let $x=\sec\theta$. Then $dx=\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-1}=\tan\theta$ and the integral equals

$$\int\cos^3\theta\,d\theta=\int(1-\sin^2\theta)\cos\theta\,d\theta=\sin\theta-\tfrac{1}{3}\sin^3\theta+C$$

(as in number 6, we handle the odd power of $\cos\theta$ with the substitution $u=\sin\theta$).

10. $x\arctan x-\frac{1}{2}\ln(1+x^2)+C$. Integrate by parts with $u=\arctan x$ and $dv=dx$.

11. $\frac{1}{3}\sec^3 x-2\sec x-\cos x+C$. The integral equals

$$\int\frac{(1-\cos^2 x)^2}{\cos^4 x}\sin x\,dx=-\int\frac{(1-u^2)^2}{u^4}\,du=-\int(u^{-4}-2u^{-2}+1)\,du$$

where $u=\cos x$.

12. $x^2+2x+\ln|2x+1|+7\ln|x-3|+C$. Long division followed by partial fractions.

13. $\left[x\ln x-x\right]_1^2=\ln 4-1$. Integrate by parts with $u=\ln x$ and $dv=dx$.

14. $\frac{1}{2}\arctan(e^{2x})+C$. Let $u=e^{2x}$.

15. $\left[-3\sqrt{4-x^2}-\arcsin(x/2)\right]_0^1=6-3\sqrt{3}-\frac{1}{6}\pi$. The integral equals

$$3\int_0^1\frac{x\,dx}{\sqrt{4-x^2}}-\int_0^1\frac{dx}{\sqrt{4-x^2}}$$

(For the first integral, let $u=4-x^2$.) Compare with number 5.

16. $\frac{1}{8}(x-\sin x)+C$. Using half-angle identities, the integrand equals

$$\tfrac{1}{4}\big(1-\cos^2(x/2)\big)=\tfrac{1}{4}\big(1-\tfrac{1}{2}(1+\cos x)\big)=\tfrac{1}{8}(1-\cos x)$$

(Alternatively, let $u=x/4$ and reduce to number 3.)

17. $-\frac{1}{3}(x^2+50)\sqrt{25-x^2}+C$ (after simplifying). The substitution $t^2=25-x^2$ is most efficient; then $x^2=25-t^2$, $x\,dx=-t\,dt$, and the integral equals

$$\int\frac{(25-t^2)(-t)}{t}\,dt=\tfrac{1}{3}t^3-25t+C=\tfrac{1}{3}(25-x^2)^{3/2}-25\sqrt{25-x^2}+C$$

Other possibilities are to let $t = 25 - x^2$ (almost as efficient) or to use the trig substitution $x = 5\sin\theta$ (not efficient). Integrating by parts also works (let $u = x^2$ and $dv = x\,dx/\sqrt{25-x^2}$), but note that for the similar integral $\int x^5\,dx/\sqrt{25-x^2}$, one would have to integrate by parts twice, whereas the same substitution $t^2 = 25 - x^2$ still works with only a bit more work.

18. $\frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2-1} + C$. Integrate by parts with $u = \operatorname{arcsec} x$ and $dv = x\,dx$.

19. $\frac{1}{4}(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x) + C$. This is example 4 on p. 312 of the text.

20. $-\dfrac{(x^2+1)^{3/2}}{3x^3} + C$. Trig substitution: let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$ and the integral equals

$$\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = -\frac{1}{3\sin^3\theta} + C$$

21. $\frac{1}{5}e^{2x}(2\cos x + \sin x) + C$. Boomerang; see example 4 on p. 307 of the text.

22. $x\arccos x - \sqrt{1-x^2} + C$. Integrate by parts with $u = \arccos x$ and $dv = dx$.

23. $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$ or $\frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$ (these are equivalent because $\tan^2 x + 1 = \sec^2 x$). Either write the integral as

$$\int (\tan^2 x + 1)\tan^3 x \sec^2 x\,dx = \int (u^2+1)u^3\,du$$

where $u = \tan x$, or write it as

$$\int \sec^3 x(\sec^2 x - 1)\sec x \tan x\,dx = \int u^3(u^2-1)\,du$$

where $u = \sec x$.

24. $2\sqrt{\sec x} + C$. The integral equals

$$\int \frac{\sqrt{\sec x}}{\sec x}\sec x\tan x\,dx = \int \frac{\sqrt{u}}{u}\,du = \int u^{-1/2}\,du$$

where $u = \sec x$. The similarity with the second solution in number 23 is because of the odd power of $\tan x$ in both integrals.

25. $2(\sqrt{x}-1)e^{\sqrt{x}} + C$. Let $t^2 = x$ (so $t = \sqrt{x}$). Then $dx = 2t\,dt$ and the integral equals $2\int te^t\,dt$. Now integrate by parts with $u = t$ and $dv = e^t\,dt$.

26. $\frac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C$. Partial fractions.

27. $\dfrac{x^3}{3(1-x^2)^{3/2}} + \dfrac{x}{\sqrt{1-x^2}} + C$. Trig substitution: let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $\sqrt{1-x^2} = \cos\theta$ and the integral equals

$$\int \sec^4\theta\,d\theta = \int (\tan^2\theta + 1)\sec^2\theta\,d\theta = \tfrac{1}{3}\tan^3\theta + \tan\theta + C$$

(as in the first solution in number 23, we handle the even power of $\sec\theta$ with the substitution $u = \tan\theta$).

28. $\frac{2}{5}\tan^5(x/2) + \frac{2}{3}\tan^3(x/2) + C$. It's convenient to first write the integral as $2\int \sec^4 t\,\tan^2 t\,dt$, where $t = x/2$. Because of the even power of $\sec t$, we write this integral as

$$2\int (\tan^2 t + 1)\tan^2 t\sec^2 t\,dt = 2\int (u^2+1)u^2\,du$$

where $u = \tan t$ (compare with numbers 23 and 27).

29. $\frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x\,\Big]_0^1 = \frac{1}{4}(\pi - 2)$. Integrate by parts with $u = \arctan x$ and $dv = x\,dx$ (compare with number 10). Long division is then needed if $v = \frac{1}{2}x^2$ is chosen as antiderivative of x; choosing $v = \frac{1}{2}x^2 + \frac{1}{2}$ instead avoids the long division (because the integral simplifies).

30. $\pi/24$. Let $u = x^2$; then $\frac{1}{2}\,du = x\,dx$ and the integral equals

$$\int_{\sqrt[4]{2}}^{\sqrt{2}} \frac{x\,dx}{x^2\sqrt{x^4-1}} = \frac{1}{2}\int_{\sqrt{2}}^{2} \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\operatorname{arcsec} u\,\Big]_{\sqrt{2}}^{2}$$

31. $\ln|\sec x + \tan x| - \sin x + C$. The integrand equals

$$\frac{1-\cos^2 x}{\cos x} = \sec x - \cos x$$

32. $x\operatorname{arcsec} x - \ln\left|x + \sqrt{x^2-1}\right| + C$. Integrate by parts with $u = \operatorname{arcsec} x$ and $dv = dx$. For the resulting integral $\int dx/\sqrt{x^2-1}$, you can use the trig substitution $x = \sec\theta$. Compare with number 18.

33. $\frac{1}{5}e^x(\sin 2x - 2\cos 2x) + C$. Boomerang; see example 4 on p. 307 of the text.

34. $\frac{1}{2}x\big(\sin(\ln x) - \cos(\ln x)\big) + C$. Let $t = \ln x$. Then $x = e^t$, $dx = e^t\,dt$, and the integral equals $\int e^t \sin t\,dt$, which is the boomerang in example 4 on p. 307 of the text.

35. $2\sqrt{x}\tan\sqrt{x} - 2\ln|\sec\sqrt{x}| + C$. Let $t^2 = x$ (so $t = \sqrt{x}$). Then $dx = 2t\,dt$ and the integral equals $2\int t\sec^2 t\,dt$. Now integrate by parts with $u = t$ and $dv = \sec^2 t\,dt$.

36. $\frac{1}{2}\tan^{-1} x + \dfrac{x}{2(x^2+1)} + C$. Trig substitution: let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \sec^2\theta$ and the integral equals

$$\int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\,d\theta = \int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta + C$$

Additional remarks

1. Integration by parts was used in numbers 1, 2, 4, 8, 10, 13, 18, 21, 22, 25, 29, 32–35. Most of these involve a polynomial (or the constant 1) times either a transcendental function (exponential, log, trig or inverse trig) or a product of transcendentals (as in 8, 21, 33). A substitution may be needed to first convert the integral to this form (as in 25, 34, 35).

2. For trig integrals (numbers 3, 6, 11, 16, 19, 23, 24, 28, 31) and trig substitutions (9, 20, 27, 36), it is very helpful to remember the strategies described in the margin of pages 312 and 313 of the text. Notice that, as number 24 shows, the strategy for odd powers of $\tan x$ (i.e., letting $u = \sec x$) works even if $\sec x$ is not actually a factor of the integrand (just multiply and divide by $\sec x$). Also notice that in number 20, the text's strategy for powers of $\tan x$ and $\sec x$ doesn't apply; that's why we converted $\sec^3\theta/\tan^4\theta$ to sines and cosines.