Cal II: Worksheet 6 (improper integrals)

Determine whether the following integrals converge or diverge. If an integral converges, give its exact value. Remember that each integral will require that you take one or more limits, *using the correct notation.* And if any *one* of these limits does not exist, you can immediately conclude that the integral must diverge (don't do more work than is necessary).

**1.** $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-1}}$ **2.** $\displaystyle\int_0^{\pi/2} \frac{dx}{\cos x}$ **3.** $\displaystyle\int_1^\infty \frac{dx}{x(x+1)}$ **4.** $\displaystyle\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

**5.** $\displaystyle\int_1^2 \frac{dx}{\sqrt{4-x^2}}$ **6.** $\displaystyle\int_0^1 \frac{e^x}{1-e^x}\,dx$ **7.** $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$ **8.** $\displaystyle\int_{-\infty}^{1/\sqrt{3}} \frac{dx}{9x^2+1}$

**9.** $\displaystyle\int_{-1}^1 \frac{4}{(1-x)^3}\,dx$ **10.** $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1-\tan x}}\,dx$ **11.** $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$ **12.** $\displaystyle\int_0^4 \frac{dx}{x^2-2x-3}$

**13.** $\displaystyle\int_0^\pi \frac{\cos x}{\sqrt{\sin x}}\,dx$ **14.** $\displaystyle\int_{-\infty}^0 x^2 e^x\,dx$ **15.** $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2-9}}$

**Answers and comments**

**1.** Converges to 6. The integral equals

$$\lim_{t\to1^+} \tfrac{3}{2}(x-1)^{2/3}\Big]_t^9 = \lim_{t\to1^+} \tfrac{3}{2}\big(4-(t-1)^{2/3}\big) = 6$$

**2.** Diverges. The integral equals

$$\lim_{t\to(\pi/2)^-} \ln|\sec x+\tan x|\Big]_0^t = \lim_{t\to(\pi/2)^-} \ln|\sec t+\tan t| = \infty$$

because $\sec t\to\infty$ and $\tan t\to\infty$ as $t\to(\pi/2)^-$.

**3.** Converges to $\ln 2$. Using partial fractions, the integral equals

$$\lim_{t\to\infty} \ln\Big|\frac{x}{x+1}\Big|\,\Big]_1^t = \lim_{t\to\infty}\Big(\ln\Big|\frac{t}{t+1}\Big| - \ln\tfrac{1}{2}\Big) = -\ln\tfrac{1}{2} = \ln 2$$

because $t/(t+1)\to1$ as $t\to\infty$.

**4.** Converges to 2. Because of the discontinuity at 0, we have to split this up into two integrals:

$$\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx + \int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$
$$= \lim_{t\to0^+} -2e^{-\sqrt{x}}\Big]_t^1 + \lim_{t\to\infty} -2e^{-\sqrt{x}}\Big]_1^t = 2-2/e+2/e$$

**5.** Converges to $\pi/3$. The integral equals

$$\lim_{t\to2^-} \sin^{-1}(x/2)\Big]_1^t = \lim_{t\to2^-}\big(\sin^{-1}(t/2)-\tfrac{1}{6}\pi\big) = \tfrac{1}{2}\pi-\tfrac{1}{6}\pi$$

**6.** Diverges. The integral equals

$$\lim_{t\to0^+} -\ln|1-e^x|\Big]_t^1 = \lim_{t\to0^+}\big(-\ln|1-e|+\ln|1-e^t|\big) = -\infty$$

**7.** Converges to 1. The integral equals

$$\lim_{t\to\infty} -\frac{1}{\ln x}\Big]_e^t = \lim_{t\to\infty}\Big(-\frac{1}{\ln t}+1\Big) = 1$$

**8.** Converges to $5\pi/18$. The integral equals

$$\lim_{t\to-\infty} \tfrac{1}{3}\tan^{-1}(3x)\Big]_t^{1/\sqrt{3}} = \lim_{t\to-\infty}\tfrac{1}{3}\big(\tfrac{1}{3}\pi-\tan^{-1}(3t)\big)$$
$$= \tfrac{1}{3}\big(\tfrac{1}{3}\pi+\tfrac{1}{2}\pi\big) = \tfrac{5}{18}\pi$$

because $\tan^{-1}(3t)\to-\pi/2$ as $t\to-\infty$.

**9.** Diverges. The integral equals

$$\lim_{t\to1^-} \frac{2}{(1-x)^2}\Big]_{-1}^t = \lim_{t\to1^-}\frac{2}{(1-t)^2}-\frac{1}{2} = \infty$$

**10.** Converges to 2. The integral equals

$$\lim_{t\to(\pi/4)^-} -2\sqrt{1-\tan x}\,\Big]_0^t = \lim_{t\to(\pi/4)^-} -2\sqrt{1-\tan t}+2 = 2$$

**11.** Converges to $\frac{1}{2}$. The integral (boomerang) equals

$$\lim_{t\to\infty} \tfrac{1}{2}e^{-x}(\sin x-\cos x)\Big]_0^t = \lim_{t\to\infty}\big(\tfrac{1}{2}e^{-t}(\sin t-\cos t)+\tfrac{1}{2}\big)$$
$$= 0+\tfrac{1}{2}$$

(As $t\to\infty$, $e^{-t}(\sin t-\cos t)\to0$ by the Squeeze Theorem.)

**12.** Diverges. Because of the discontinuity at 3, we have to split this up into two integrals

$$\int_0^3 \frac{dx}{x^2-2x-3} + \int_3^4 \frac{dx}{x^2-2x-3}$$

Using partial fractions, the first integral equals

$$\lim_{t\to3^-} \tfrac{1}{4}\big(\ln|x-3|-\ln|x+1|\big)\Big]_0^t = -\infty$$

**13.** Converges to 0. Because of the discontinuities at 0 and $\pi$, we have to split this up into two integrals:

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx + \int_{\pi/2}^\pi \frac{\cos x}{\sqrt{\sin x}}\,dx$$
$$= \lim_{t\to0^+} 2\sqrt{\sin x}\,\Big]_t^{\pi/2} + \lim_{t\to\pi^-} 2\sqrt{\sin x}\,\Big]_{\pi/2}^t = 2-2$$

**14.** Converges to 2. Integrating by parts (twice), the integral equals

$$\lim_{t\to-\infty} (x^2-2x+2)e^x\Big]_t^0 = 2$$

**15.** Converges to $\pi/6$. Because of the discontinuity at 3, we have to split this up into two integrals:

$$\int_3^6 \frac{dx}{x\sqrt{x^2-9}} + \int_6^\infty \frac{dx}{x\sqrt{x^2-9}}$$
$$= \lim_{t\to3^+} \tfrac{1}{3}\sec^{-1}(x/3)\Big]_t^6 + \lim_{t\to\infty}\tfrac{1}{3}\sec^{-1}(x/3)\Big]_6^t$$
$$= \tfrac{1}{3}\big(\tfrac{1}{3}\pi-0\big)+\tfrac{1}{3}\big(\tfrac{1}{2}\pi-\tfrac{1}{3}\pi\big)$$