(1–9) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

1. $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$ **2.** $\sum_{n=1}^{\infty} \frac{4^{n+2}}{7^n}$

3. $\sum_{n=1}^{\infty} \frac{6}{(3n+2)(3n-1)}$ **4.** $\sum_{k=1}^{\infty} \frac{2}{k^2+2k}$

5. $\sum_{n=0}^{\infty} \frac{1+2^n}{3^n}$ **6.** $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

7. $\sum_{n=0}^{\infty} \frac{5^n-2^n}{7^n}$ **8.** $\sum_{n=1}^{\infty} \left[\frac{(-3)^n}{2^{n+1}} + \frac{1}{n(n+1)}\right]$

9. $\sum_{n=1}^{\infty} \left[\tan^{-1} n - \tan^{-1}(n+1)\right]$

(10–41) Determine whether the series converges or diverges. State which test you are using and verify that the conditions for using it are satisfied.

10. $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$ **11.** $\sum_{n=1}^{\infty} \left(\frac{n^2+3}{2n^2+1}\right)^n$

12. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ **13.** $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

14. $\sum_{n=2}^{\infty} \left[\frac{n}{\ln n} + \left(\frac{2}{3}\right)^n\right]$ **15.** $\sum_{n=1}^{\infty} \frac{\sqrt{2n+3}}{5n^2-1}$

16. $\sum_{n=1}^{\infty} \frac{ne^n}{(2n)!}$ **17.** $\sum_{n=1}^{\infty} \frac{3n^2+5}{5n^2+3}$

18. $\sum_{n=1}^{\infty} \left[\frac{1}{n^3} - \frac{1}{3n}\right]$ **19.** $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

20. $\sum_{n=1}^{\infty} \left(\frac{1}{3} + \frac{1}{2n}\right)^n$ **21.** $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n}}$

22. $\sum_{n=2}^{\infty} \frac{n^2-3n+1}{2n^3+5n-7}$ **23.** $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

24. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **25.** $\sum_{n=1}^{\infty} \frac{3^{2n}}{n^n}$

26. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ **27.** $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ **29.** $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$

30. $\sum_{n=1}^{\infty} \frac{3-\sin n}{2n}$ **31.** $\sum_{n=1}^{\infty} \frac{1}{2n+\ln n}$

32. $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^5+2n^2}}$ **33.** $\sum_{n=0}^{\infty} e^{-n}$

34. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n-1}}$ **35.** $\sum_{n=1}^{\infty} \left(2 - \frac{n}{n^2+4}\right)$

36. $\sum_{n=1}^{\infty} \frac{\sec^2 n}{\sqrt[3]{n}}$ **37.** $\sum_{n=1}^{\infty} \frac{n^3-2}{n^5-2n^2+6}$

38. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ **39.** $\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt[3]{n^4}}$

40. $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n-2}\right)^n$ **41.** $\sum_{n=1}^{\infty} \frac{n^n\, n!}{(2n)!}$

Solution outlines

1. Telescoping series:

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

so the nth partial sum is

$$s_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

which tends to 1 as $n \to \infty$, i.e., the series converges to 1.

2. Geometric series with common ratio $r = \frac{4}{7}$ and first term $a = 4^{1+2}/7^1 = 64/7$. Since $|r| < 1$, the series converges to

$$\frac{a}{1-r} = \frac{\frac{64}{7}}{1-\frac{4}{7}} = \frac{64}{3}$$

3. Telescoping series:

$$\frac{6}{(3n+2)(3n-1)} = \frac{2}{3n-1} - \frac{2}{3n+2}$$

so the nth partial sum is

$$s_n = \left(1-\frac{2}{5}\right) + \left(\frac{2}{5}-\frac{2}{8}\right) + \cdots + \left(\frac{2}{3n-1} - \frac{2}{3n+2}\right)$$
$$= 1 - \frac{2}{3n+2} \longrightarrow 1 \quad \text{as } n \to \infty$$

i.e., the series converges to 1.

4. Telescoping series:

$$\frac{2}{k^2+2k} = \frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2}$$

so the nth partial sum, for $n \geqslant 3$, is

$$s_n = \left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right)$$
$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \longrightarrow \frac{3}{2} \quad \text{as } n \to \infty$$

i.e., the series converges to $\frac{3}{2}$.

5. This is the sum of two convergent geometric series, and it converges to

$$\sum_{n=0}^{\infty} (\tfrac{1}{3})^n + \sum_{n=0}^{\infty} (\tfrac{2}{3})^n = \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{2}{3}} = \tfrac{3}{2} + 3 = \tfrac{9}{2}$$

6. Since $a_n = \ln n - \ln(n+1)$, this is a telescoping series whose nth partial sum is

$$s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + \big(\ln n - \ln(n+1)\big)$$
$$= -\ln(n+1) \longrightarrow -\infty \quad \text{as } n \to \infty$$

i.e., the series diverges.

7. This is the difference of two convergent geometric series, and it converges to

$$\sum_{n=0}^{\infty}(\tfrac{5}{7})^n - \sum_{n=0}^{\infty}(\tfrac{2}{7})^n = \frac{1}{1-\frac{5}{7}} - \frac{1}{1-\frac{2}{7}} = \tfrac{7}{2} - \tfrac{7}{5} = \tfrac{21}{10}$$

8. Diverges by the divergence test because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\tfrac{1}{2}(-1)^n\left(\tfrac{3}{2}\right)^n + \frac{1}{n(n+1)}\right]$$

doesn't exist: as $n \to \infty$, the first term oscillates wildly (since $\frac{3}{2} > 1$) while the second tends to 0.

9. This is a telescoping series whose nth partial sum is

$$s_n = \tan^{-1}(1) - \tan^{-1}(n+1) = \tfrac{1}{4}\pi - \tan^{-1}(n+1)$$
$$\longrightarrow \tfrac{1}{4}\pi - \tfrac{1}{2}\pi = -\tfrac{1}{4}\pi \quad \text{as } n \to \infty$$

i.e., the series converges to $-\frac{1}{4}\pi$.

10. Diverges by the divergence test: $\lim_{n\to\infty} 5^n/n^5 = \infty$ doesn't exist (5^n grows exponentially while n^5 only grows polynomially). Alternatively, the ratio test gives

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)^5}\cdot\frac{n^5}{5^n} = 5\Big(\frac{n}{n+1}\Big)^5 \longrightarrow 5 \quad \text{as } n \to \infty$$

Since the limit is greater than 1, the series diverges.

11. Converges by the root test:

$$\sqrt[n]{a_n} = \frac{n^2+3}{2n^2+1} = \frac{1+3/n^2}{2+1/n^2} \longrightarrow \tfrac{1}{2} < 1 \quad \text{as } n \to \infty$$

12. Diverges by the integral test: $f(x) = 1/x\sqrt{\ln x}$ is a continuous, positive, decreasing function on $[2,\infty)$ and

$$\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} 2\sqrt{\ln x}\,\Big]_2^t = \lim_{t\to\infty} 2\big(\sqrt{\ln t} - \sqrt{\ln 2}\big) = \infty$$

13. Diverges by the divergence test:

$$a_n = \Big(1+\frac{1}{n}\Big)^n \longrightarrow e \neq 0 \quad \text{as } n \to \infty$$

14. Diverges by the divergence test: as $n \to \infty$, $n/\ln n \to \infty$ by L'Hospital's Rule, while $(\frac{2}{3})^n \to 0$ since $\left|\frac{2}{3}\right| < 1$; it follows that $\lim_{n\to\infty} a_n = \infty$ doesn't exist.

15. Since

$$a_n = \frac{\sqrt{2n+3}}{5n^2-1} = \frac{\sqrt{n}\sqrt{2+3/n}}{n^2(5-1/n^2)} = \frac{1}{n^{3/2}}\cdot\frac{\sqrt{2+3/n}}{5-1/n^2}$$

use the limit comparison test with the convergent p-series $\sum b_n = \sum 1/n^{\frac{3}{2}}$:

$$\frac{a_n}{b_n} = \frac{\sqrt{2+3/n}}{5-1/n^2} \longrightarrow \frac{\sqrt{2}}{5} \neq 0 \quad \text{as } n \to \infty$$

so the given series converges.

16. Converges by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)e^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{ne^n} = \frac{(n+1)e}{n(2n+2)(2n+1)}$$
$$= \frac{e}{2n(2n+1)} \longrightarrow 0 < 1 \quad \text{as } n \to \infty$$

17. Diverges by the divergence test:

$$a_n = \frac{3n^2+5}{5n^2+3} = \frac{3+5/n^2}{5+3/n^2} \longrightarrow \tfrac{3}{5} \neq 0 \quad \text{as } n \to \infty$$

18. The series $\sum 1/(3n) = \frac{1}{3}\sum 1/n$ diverges (harmonic series). Since the series $\sum 1/n^3$ converges (it's a p-series with $p = 3 > 1$), it follows that the given series must also diverge.

19. Converges by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\big[(n+1)!\big]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)}$$
$$= \frac{n+1}{2(2n+1)} = \frac{1+1/n}{4+2/n} \longrightarrow \tfrac{1}{4} < 1 \quad \text{as } n \to \infty$$

20. Converges by the root test:

$$\sqrt[n]{a_n} = \frac{1}{3} + \frac{1}{2n} \longrightarrow \tfrac{1}{3} < 1 \quad \text{as } n \to \infty$$

21. Diverges by the nth tem test:

$$a_n = \sqrt{\frac{n+1}{n}} = \sqrt{1+\frac{1}{n}} \longrightarrow 1 \neq 0 \quad \text{as } n \to \infty$$

22. Since

$$a_n = \frac{n^2-3n+1}{2n^3+5n-7} = \frac{1}{n}\cdot\frac{1-3/n+1/n^2}{2+5/n^2-7/n^3}$$

use the limit comparison test with the divergent harmonic series $\sum b_n = \sum 1/n$:

$$\frac{a_n}{b_n} = \frac{1-3/n+1/n^2}{2+5/n^2-7/n^3} \longrightarrow \tfrac{1}{2} \neq 0 \quad \text{as } n \to \infty$$

so the given series diverges.

23. Diverges by the root test:

$$\sqrt[n]{a_n} = \Big(1+\frac{1}{n}\Big)^n \longrightarrow e > 1 \quad \text{as } n \to \infty$$

24. Diverges by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \frac{(n+1)^{n+1}}{(n+1)\,n^n} = \frac{(n+1)^n}{n^n}$$
$$= \Big(1+\frac{1}{n}\Big)^n \longrightarrow e > 1 \quad \text{as } n \to \infty$$

25. Converges by the root test:

$$\sqrt[n]{a_n} = \Big(\frac{3^{2n}}{n^n}\Big)^{1/n} = \Big(\frac{9^n}{n^n}\Big)^{1/n} = \frac{9}{n} \longrightarrow 0 < 1 \quad \text{as } n \to \infty$$

Alternatively,

$$a_n = \frac{3^{2n}}{n^n} = \Big(\frac{9}{n}\Big)^n \leqslant \Big(\frac{9}{10}\Big)^n = b_n \quad \text{for } n \geqslant 10$$

and so the given series converges by direct comparison with the convergent geometric series $\sum b_n = \sum(\frac{9}{10})^n$.

26. Since $\ln x > 1$ when $x > e$,

$$a_n = \frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}} = b_n \quad \text{for } n \geqslant 3$$

so the given series diverges by direct comparison with the divergent p-series $\sum b_n = \sum 1/\sqrt{n}$. (Or use the integral test, but this requires integrating by parts—more work.)

27. Converges by the integral test: $f(x) = \ln x/x^2$ is a continuous, positive, decreasing function on $[2, \infty)$ and

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} -\frac{1+\ln x}{x}\Big]_2^t = \tfrac{1}{2}(1+\ln 2) < \infty$$

(using integration by parts and L'Hospital's Rule). Alternatively, since $\lim_{x\to\infty} \ln x/\sqrt{x} = 0$ (by L'Hospital's Rule), $\ln n < \sqrt{n}$ for all sufficiently large n, and so

$$a_n = \frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} = b_n$$

The given series then converges by direct comparison with the convergent p-series $\sum b_n = \sum 1/n^{\frac{3}{2}}$.

28. Since

$$a_n = \frac{e^{1/n}}{n^2} \leqslant \frac{e}{n^2} = e \cdot \frac{1}{n^2} \quad \text{for } n \geqslant 1$$

the series converges by direct comparison with the convergent p-series $\sum 1/n^2$. (Alternatively, use the limit comparison test with the same p-series, or use the integral test.)

29. Since

$$a_n = \frac{n!}{(n+2)!} = \frac{1}{(n+2)(n+1)} < \frac{1}{n^2} = b_n \quad \text{for all } n \geqslant 1$$

the series converges by direct comparison with the convergent p-series $\sum b_n = \sum 1/n^2$. (Alternatively, use the limit comparison test with the same p-series.)

30. Since $\sin n \leqslant 1$ for all n, $-\sin n \geqslant -1$, and so $3 - \sin n \geqslant 2$. It follows that

$$a_n = \frac{3-\sin n}{2n} \geqslant \frac{2}{2n} = \frac{1}{n} = b_n \quad \text{for all } n \geqslant 1$$

The series therefore diverges by direct comparison with the (divergent) harmonic series $\sum b_n = \sum 1/n$.

31. The series diverges: since

$$a_n = \frac{1}{2n + \ln n} = \frac{1}{n} \cdot \frac{1}{2 + \ln n/n}$$

we can use the limit comparison test with the (divergent) harmonic series $\sum b_n = \sum 1/n$:

$$\frac{a_n}{b_n} = \frac{1}{2 + \ln n/n} \longrightarrow \tfrac{1}{2} \neq 0 \quad \text{as } n \to \infty$$

(because $\lim_{x\to\infty} \ln x/x = 0$ by L'Hospital's Rule).

32. Since

$$a_n = \frac{n^2+1}{\sqrt{n^5+2n^2}} = \frac{n^2(1+1/n^2)}{\sqrt{n^5}\sqrt{1+2/n^3}} = \frac{1}{\sqrt{n}} \cdot \frac{1+1/n^2}{\sqrt{1+2/n^3}}$$

we can use the limit comparison test with the divergent p-series $\sum b_n = \sum 1/\sqrt{n}$:

$$\frac{a_n}{b_n} = \frac{1+1/n^2}{\sqrt{1+2/n^3}} \longrightarrow 1 \neq 0 \quad \text{as } n \to \infty$$

so the given series diverges.

33. Converges since it is a geometric series with common ratio $r = 1/e$ and $|r| < 1$.

34. Since

$$a_n = \frac{1}{n\sqrt{n-1}} = \frac{1}{n\sqrt{n}\sqrt{1-1/n}} = \frac{1}{n^{3/2}} \cdot \frac{1}{\sqrt{1-1/n}}$$

use the limit comparison test with the convergent p-series $\sum b_n = \sum 1/n^{\frac{3}{2}}$:

$$\frac{a_n}{b_n} = \frac{1}{\sqrt{1-1/n}} \longrightarrow 1 \neq 0 \quad \text{as } n \to \infty$$

so the given series converges.

35. Diverges by the divergence test:

$$a_n = 2 - \frac{n}{n^2+4} = 2 - \frac{1}{n(1+4/n^2)} \longrightarrow 2 \neq 0 \quad \text{as } n \to \infty$$

36. Since $\cos^2 n \leqslant 1$ for all n, $\sec^2 n \geqslant 1$, and so

$$a_n = \frac{\sec^2 n}{\sqrt[3]{n}} \geqslant \frac{1}{\sqrt[3]{n}} = b_n \quad \text{for all } n \geqslant 1$$

The series therefore diverges by direct comparison with the divergent p-series $\sum b_n = \sum 1/n^{\frac{1}{3}}$.

37. Since

$$a_n = \frac{n^3(1-2/n^3)}{n^5(1-2/n^3+6/n^5)} = \frac{1}{n^2} \cdot \frac{1-2/n^3}{1-2/n^3+6/n^5}$$

use the limit comparison test with the convergent p-series $\sum b_n = \sum 1/n^2$:

$$\frac{a_n}{b_n} = \frac{1-2/n^3}{1-2/n^3+6/n^5} \longrightarrow 1 \neq 0 \quad \text{as } n \to \infty$$

so the given series converges.

38. Converges by the integral test: $f(x) = 1/x(\ln x)^{\frac{3}{2}}$ is a continuous, positive, decreasing function on $[2, \infty)$ and

$$\int_2^\infty \frac{dx}{x(\ln x)^{3/2}} = \lim_{t\to\infty} -\frac{2}{\sqrt{\ln x}}\Big]_2^t = \frac{2}{\sqrt{\ln 2}} < \infty$$

39. Since $\cos^2 n \leqslant 1$ for all n,

$$a_n = \frac{\cos^2 n}{\sqrt[3]{n^4}} \leqslant \frac{1}{\sqrt[3]{n^4}} = b_n \quad \text{for all } n \geqslant 1$$

The series therefore converges by direct comparison with the convergent p-series $\sum b_n = \sum 1/n^{\frac{4}{3}}$.

40. Converges by the root test:

$$\sqrt[n]{a_n} = \frac{n+1}{3n-2} = \frac{1+1/n}{3-2/n} \longrightarrow \tfrac{1}{3} < 1 \quad \text{as } n \to \infty$$

41. Converges by the ratio test:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n^n\,n!} = \frac{(n+1)^{n+1}(n+1)}{(2n+2)(2n+1)\,n^n} \\ &= \frac{(n+1)^n}{n^n} \cdot \frac{(n+1)^2}{2(n+1)(2n+1)} \\ &= \left(1+\frac{1}{n}\right)^n \frac{n+1}{4n+2} \longrightarrow \frac{e}{4} < 1 \quad \text{as } n \to \infty\end{aligned}$$