(1–15) Determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Justify your answer.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^5}{(n+3)!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{(2n+1)^2}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$

4. $\displaystyle\sum_{n=0}^{\infty} \cos(n\pi)\, e^{-n}$

5. $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt{n}}{\ln n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\sec n\pi}{n}$

8. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

9. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{4n^2+5n}}$

10. $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n}$

11. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n!}{(2n)!}$

12. $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{1}{(\ln n)^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n+2}}$

14. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\, n\sin(1/n)$

15. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+5}$

(16–26) Find the radius of convergence and interval of convergence of the power series.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\, x^n}{n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\, 5^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^3\, 4^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-4)^n}{4^n\sqrt{n+1}}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+4)^n}{5^n\,(2n+1)}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n\, n^2}$

23. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(x-1)^n}{n}$

24. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n\ln n}$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{n\, 2^n(x+1)^n}{n+1}$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+1}}$

(27–31) Find the Maclaurin series for $f(x)$ and its radius of convergence (write the first four non-zero terms of the series explicitly, then find the nth term and express the series in sigma notation).

27. $f(x) = \ln(x+1)$

28. $f(x) = e^{-x}$

29. $f(x) = e^{x/3}$

30. $f(x) = \cos 2x$

31. $f(x) = \sin(x/2)$

32. Find the Taylor series for $f(x) = \sqrt{x}$ centred at $a = 1$ (write the first four non-zero terms of the series explicitly, then find the nth term and express the series in sigma notation).

Answers

1. AC. Ratio test: $|a_{n+1}/a_n| \to 0$ as $n \to \infty$.

2. CC. $\sum |a_n|$ diverges by limit comparison with $\sum 1/n$, but $\sum a_n$ converges by the alternating series test.

3. CC. $\sum |a_n|$ diverges by the integral test, but $\sum a_n$ converges by the alternating series test.

4. AC. $\sum |a_n|$ is geometric with $r = 1/e$.

5. D. Divergence test: $\sqrt{n}/\ln n \to \infty$ as $n \to \infty$ by L'Hospital's Rule.

6. AC. $\sum |a_n|$ converges by direct comparison with the p-series $\sum 1/n^{\frac{3}{2}}$.

7. CC. Alternating harmonic series.

8. AC. $\sum |a_n|$ converges by the integral test.

9. D. Divergence test: $n/\sqrt{4n^2+5n} \to \frac{1}{2}$ as $n \to \infty$.

10. CC. $\sum |a_n|$ diverges by the integral test (or by direct comparison with $\sum 1/n$), but $\sum a_n$ converges by the alternating series test.

11. AC. Ratio test: $|a_{n+1}/a_n| \to 0$ as $n \to \infty$.

12. AC. $\sum |a_n|$ converges by the root test.

13. CC. $\sum |a_n|$ diverges (p-series with $p = \frac{1}{2} < 1$), but $\sum a_n$ converges by the alternating series test.

14. D. Divergence test: $n\sin(1/n) \to 1$ as $n \to \infty$.

15. CC. $\sum |a_n|$ diverges by limit comparison with the p-series $\sum 1/\sqrt{n}$, but $\sum a_n$ converges by the alternating series test.

16. $R = 3$, $(-5, 1)$

17. $R = \frac{1}{2}$, $[-\frac{1}{2}, \frac{1}{2})$

18. $R = 5$, $(-4, 6]$

19. $R = 4$, $[-7, 1]$

20. $R = 4$, $(0, 8]$

21. $R = 5$, $(-9, 1]$

22. $R = 2$, $[-2, 2]$

23. $R = 1$, $(0, 2]$

24. $R = 1$, $[-1, 1)$

25. $R = \frac{1}{2}$, $(-\frac{3}{2}, -\frac{1}{2})$

26. $R = 1$, $[-1, 1)$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, $R = 1$.

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$, $R = \infty$.

29. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n\, n!}$, $R = \infty$.

30. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{4^n x^{2n}}{(2n)!}$, $R = \infty$.

31. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^{2n+1}\,(2n+1)!}$, $R = \infty$.

32. $\displaystyle 1 + \tfrac{1}{2}(x-1) + \sum_{n=2}^{\infty}(-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n\, n!}(x-1)^n$