Math 201-DAC-AB **Homework 1** **Due Thurs Feb 6 (at start of class)**

Winter 2020

Answers

1. Consider a similar game to 21 that we'll call 52. In this game, the players each take turns naming and adding numbers from 1 to 5, where the winner is the player who reaches 52. What's the winning strategy? And for which player? What if we want to reach the number 53? or 54? Does the answer change? Why or why not?

> **Answer:** In 21, the important number was 4. In this case it's 6. This means whoever gets to $52 - 6 = 46$ first wins. Continuing in that logic, we just keep subtracting 6 until we can't anymore and we reach our target number. So the winner is the first person to reach that target. In 52, that number is 4. So Player 1 (P1) wins by simply playing 4 and P1 can force the win. In 53, P1 can reach 5 first. In 54, no matter what move P1 starts with, P2 can reach 6 first.
>
> In all cases, once you've reached the target number, the winning strategy is to make sure the next number you name has the property that when added to your opponent's last named number, the total is 6. This ensures that after each round, your total will be 6 more than your previous total.

2. These questions all deal with tic-tac-toe (regular or inverse). For the sake of simplicity, we assume X goes first.

   (a) In inverse tic-tac-toe, given the position below, what are the best move(s) for player 2 (if any)? Why?

X	O	X

> **Answer:** So long as O avoids the bottom corners, O will win the game. Logically, the reason the corners are bad for O is because X doesn't want to go there anyway. If O takes the corner, X simply takes the spot above it and could force a draw. Any other move for O leaves enough openings for X to get himself into trouble, allowing O to force a win.

   (b) In inverse tic-tac-toe, given the position below, what are the best moves for player 1 (if any)? Why?

X	O	
	X	
		O

> **Answer:** X must go to the centre bottom. Any other move will allow O to win. Logically, this makes sense because it's the only move that doesn't create a spot where X could complete a line.

(c) Counting for rotational symmetry, how many different configurations are there where player 1 wins regular tic-tac-toe on her third move?

Answer: Accounting for rotational symmetry, there are only three ways for X to win: top row, middle row, or diagonally.

X	X	X

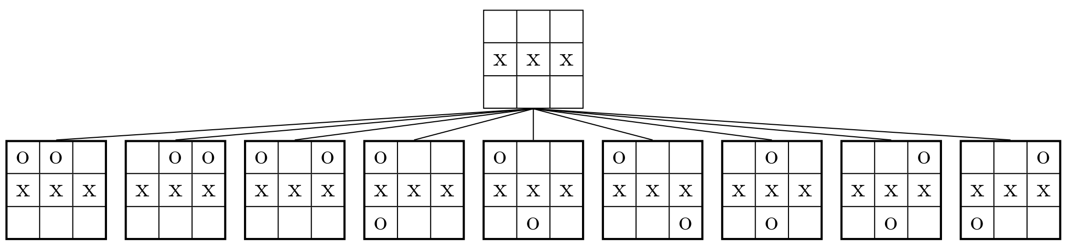
X	X	X

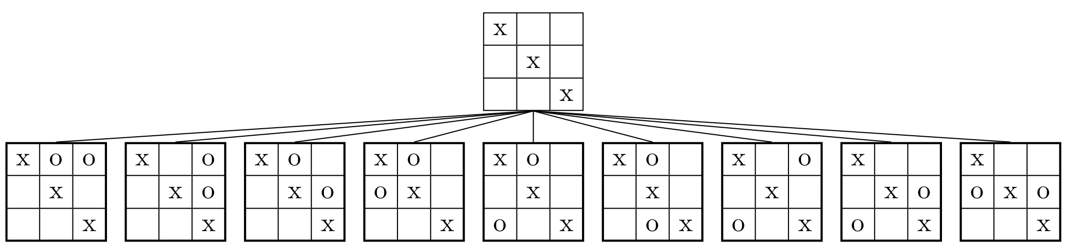
X		
	X	
		X

Since X wins in 3 moves, there will be two O's on the board.

In the first case (Top row), there are simply $\binom{6}{2} = 15$ possible places for the two O's to go.

The other two cases are trickier because they look the same after a 180° rotation. It means we have to be careful about double-counting. So it's easier just to enumerate them all.





In BOTH cases, there are 9 possible configurations. That's not an accident. If you rotate the boards in the diagonal case counterclockwise by 45°, you should be able to see why. In any case, we have a total of $15 + 9 + 9 = 33$ configurations.

(d) Let's consider a new variation of the game. Gravity tic-tac-toe is a combination of tic-tac-toe and connect-four. Essentially you can only choose which column to place an X or Of, and it sinks to the bottom-most available spot. You still want to get three-in-a-row to win.

Give the best strategies for player 1 and the best strategy for player 2. Justify your answer. The more clear your answer, the better. Points will be taken off for incomplete or confusing explanations.

Answer: The basic strategy is simple. As in regular tic-tac-toe, perfect play will end in a draw. Moreover, the best spot is the centre followed by the corners. So X should start on one of the sides (say the left). Then O should respond by taking the right side (any other move will lose).

If X now goes in the middle, then O can force all subsequent moves by going right, followed by middle. This will end in a draw. If X takes either side, O just plays right on top of that X . From there, it's clear that O can force a draw. When X goes middle, make sure to go middle as well, and take the centre. Otherwise, don't do anything that loses immediately.

3. Alice, Bob, Cheryl, Daniel and Elliott need to line up for a photograph. However, Alice and Bob hate each other and refuse to stand next to each other. How many different ways can we arrange the photograph without upsetting Alice and Bob?

Answer: It's quicker to instead consider the exact opposite situation, namely all the arrangements where Alice and Bob are next to each other. Then we can see them as one unit AB. And we just have to look at the permutations of AB,C,D and E.

There are of $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ such combinations.

However, if Alice and Bob are next to each other, Alice might be to the left or she may be to the right. So we have to double that number, giving us $2 \cdot 24 = 48$ possible arrangements with Alice next to Bob.

Now we subtract that from the total number of permutations of 5 people, which is $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

This leave us $120 - 48 = 72$ possible arrangements that keep Alice and Bob apart.