201-NYB-05 - Calculus 2

MODELING WITH DIFFERENTIAL EQUATIONS

1. **Mixing problem.** Consider a tank initially containing 100 liters of water with 20 kg of salt uniformly dissolved in the water. Suppose a solution with 2kg/l of salt flows into the tank at a rate of 5 l/min. The solution in the tank is kept thoroughly mixed and drains from the tank at the same rate (5 l/min).

   (a) Find how much salt is in the tank as a function of time.

   (b) What will happen to the amount of salt after a long time has elapsed?

2. **Air resistance.** Consider the velocity of a falling object as a function of time. If we ignore friction, then we can apply Newton's second law ($F = ma$) to obtain the following differential equation

   $$m\frac{dv}{dt} = -mg$$

   where the left hand side is ma and the right side is the force due to gravity. In this model, you would find a velocity that keeps increasing linearly.

   However, it is known that air resistance causes a *drag force*, which is proportional to the speed of the object (the faster the object, the stronger the drag force).

   (a) How can we modify the above differential equation to account for the drag force?

   (b) Suppose an object of mass 5 kg is released from rest, from a position high above the ground. Assume that the force due to air resistance is proportional to the velocity of the object with proportionality constant $k = 50$ N-s/m. What will happen to the object after a long time has elapsed (assuming it doesn't hit the ground)?

3. **Newton's Law of Cooling.** Suppose you have just poured a cup of coffee with temperature with temperature 95^oC in a room where the temperature is a stable 20^oC. After waiting 1 minute, the temperature of the coffee has come down to 80^oC.

   Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. Express this law as a differential equation, and solve it in this case to find the temperature of your coffee as a function of time. What will happen to the temperature of your coffee after a long time has elapsed?

4. A chemical compound is administered intravenously into your bloodstream at a constant rate r. At the same time, this chemical is eliminated by your body at a rate proportional to the concentration at that time. Write a model for the concentration of the chemical as a function of time and solve it. What happens to the concentration as time elapses?

SOLUTION FOR #3

$T(t)$ = temperature of coffee after t minutes

$$\begin{cases} \frac{dT}{dt} = k(T-20) \\ T(0) = 95 \\ T(1) = 80 \end{cases}$$

- $(T-20)$ ← difference between the coffee's temperature and the ambient temperature
- k ↑ proportionality constant
- $T(0) = 95$ ← initial temperature
- $T(1) = 80$ ← after 1 minute

⇒ separable differential equation

$$\frac{dT}{T-20} = k\,dt \Rightarrow \int \frac{1}{T-20}\,dT = \int k\,dt$$

$$\Rightarrow \ln|T-20| = kt + C$$

plug-in extra conditions to determine k and C

$$\ln|95-20| = k\cdot 0 + C \Rightarrow C = \ln(75)$$

$$\ln|80-20| = k\cdot 1 + C \Rightarrow k = \ln(60) - \ln(75) = \ln(60/75) = \ln(4/5)$$

$$\Rightarrow \ln|T-20| = \ln(4/5)t + \ln(75)$$

$$\Rightarrow |T-20| = e^{\ln(4/5)t + \ln(75)} = e^{\ln((4/5)^t) + \ln(75)}$$

$$\Rightarrow |T-20| = (75)(4/5)^t$$

$$T - 20 = \pm 75\left(\frac{4}{5}\right)^t = 75\left(\frac{4}{5}\right)^t$$

$$\Rightarrow \boxed{T(t) = 75\left(\frac{4}{5}\right)^t + 20}$$

note that

$$\lim_{t\to\infty} T(t) = 75\underbrace{\left(\frac{4}{5}\right)^{\infty}}_{\to 0} + 20 = \underline{\underline{20°C}}$$

SOLUTION FOR #4

$y(t)$ = concentration of chemical as a function of time

$$\frac{dy}{dt} = r - ky \qquad y(0) = 0$$

r ↑ rate in (constant)

ky ↑ rate out (proportional to y)

$y(0)=0$ ← initial concentration, let's say there is no chemical to start with

Separable:

$$\frac{dy}{r-ky} = dt$$

$$\int \frac{dy}{r-ky} = \int dt$$

$$-\frac{1}{k}\ln|r-ky| = t + C$$

$$\ln|r-ky| = -kt - kC$$

initial condition: $y=0, t=0$

$$\ln(r) = -kC \Rightarrow C = -\frac{1}{k}\ln(r)$$

$$|r-ky| = e^{-kt+\ln(r)} = re^{-kt}$$

$$r-ky = \pm re^{-kt}$$

$$\Rightarrow y = \frac{re^{-kt}-r}{-k}$$

$$\boxed{y(t) = \frac{r}{k}(1-e^{-kt})}$$

$$\lim_{t\to\infty} y(t) = \frac{r}{k}(1-e^{-\infty}) = \boxed{\frac{r}{k}}$$

Word problems on separable differential equations

1. The rate of increase of the population P of a village is proportional to the population size. In 2004 the population was 2500 and in 2006 it was 3000. In what year will the population reach 4320?

2. The rate of decay of a radioactive substance is proportional to the amount $N(t)$ of substance remaining at time t. If initially the amount of substance is 10 g and if 80% of the initial amount remains after 2 hours, find the amount $N(t)$ of radioactive substance at time t.

3. The rate of decomposition of a substance is proportional to the amount N of substance present at time t. If 70% of the initial amount of substance has decomposed after 4 hours, find the amount of substance that remains after 8 hours if initially the amount of substance was 120 g.

4. The rate of increase of the population P of a town is proportional to the time t (the number of years after 1980) and inversely proportional to the population size P. In 1980 the population was 10,000 and in 1990 it was 20,000. In what year will the population be 52,000?

5. The rate of increase of the population P of a city is proportional to the population size P and inversely proportional to the time t (the number of years after 1965). In 1966 the population was 800,000. By 1974 it had grown by 6%. What was the population of the city in 1983?

6. The sales volume S of a company (in millions of dollars) is increasing at a rate inversely proportional to the square root of time t (in years). At present, the sales volume is $40 million. The company predicts that in one year, the sales volume will be $50 million. Find the sales volume in 4 years.

7. The sales volume S of a company (in millions of dollars) is increasing at a rate proportional to the product of the square of both the sales volume S and the time t (in years). The company started with a sales volume of $100 million. Three years later, the sales volume reached $120 million. When will the sales volume reach $165 million?

8. A metal company's production N is increasing at a rate proportional to the product of the number N of units and the time t in years. The company production is presently 400 units. In 2 years, 1200 units are expected. What will be the production in 4 years?

9. A car is bought for $36,000. Its value V is depreciating at a rate proportional to the present value V. After 2 years, the vehicle is worth $18,000. What is the value of the car after 4 years?

10. A piece of machinery is worth $1600. Its value V is depreciating at a rate proportional to the square root of its value V. The piece of machinery will be worth $900 in two years. When will the piece of machinery be worth $625?

11. A rumor starts in a population of 10,000. The rumor spreads at a rate proportional to the number of people who at time t have not heard the rumor. Initially, 25 people have heard the rumor; at the end of 3 weeks, 6675 people have heard it. How many people will have heard the rumor after 6 weeks?

12. The production of N units is increasing at a rate proportional to the product of the number of units N and the time t in years. Initially 100 units are produced and after one year 250 units. When will production reach 500 units?

13. A piece of furniture is worth $2500. Its value V is depreciating at a rate proportional to the square of its value $V(t)$ at time t in years. The piece of furniture will be worth $1500 in two years. How much will it be worth in 3 years?

14. The rate of increase of the population P of a small city is proportional to the time in years and inversely proportional to the population size P. Initially, the population is 10,000 and in 10 years it will be 20,000. How long will it take for the population to reach 41,000?

15. A rumor starts in a population of 1,000. The rumor spreads at a rate proportional to the time t in days and inversely proportional to the number of people N who have heard the rumor. Initially, a single person starts the rumor; at the end of 5 days, 100 people have heard it. How many people will have heard the rumor after 25 days?

16. The production of N units is increasing at a rate proportional to the square of the time t in years. Initially, 1300 units are produced and, after one year, 2100 units. How long will it take for the production to reach 22,900 units?

17. A company's production is increasing at a rate proportional to the product of the number N of units and the square of the time t in years. Initially, 8 units are produced. In one year, 16 units are expected. After 2 years, what will be the production?

18. A piece of furniture is worth $2500. Its value V is depreciating at a rate proportional to the value and inversely proportional to the square root of the time t in years. The piece of furniture will be worth $1600 in four years. How long will it take for it to be worth $1280?

19. The rate of increase of the population P of a city is proportional to the product of the population size P and the square of time t in years. Initially, the population is 100,000 and in 3 years it is 300,000. How long will it take for the population to be 1,200,000?

20. A company's production is increasing at a rate proportional to the number N of units produced and inversely proportional to the square of the time t in years. In the first year, 1000 units are produced. In the long run (i.e., as $t \to \infty$), production is expected to level off at 5000 units. What will be the production after 5 years?

1. $dP/dt = kP$, where t is the number of years after 2004, $P(0) = 2500$, $P(2) = 3000$.
$P = Ce^{kt}$, $C = 2500$, $k = \frac{1}{2}\ln(1.2)$, so $P = 2500(1.2)^{t/2}$.
When $P = 4320$, $t = 2\ln(1.728)/\ln(1.2) = 6$ years (the year 2010).

2. $dN/dt = kN$, $N(0) = 10$, $N(2) = 8$.
$N = Ce^{kt}$, $C = 10$, $k = \frac{1}{2}\ln(0.8)$, so $N = 10(0.8)^{t/2}$.

3. $dN/dt = kN$, $N(0) = 120$, $N(4) = 36$.
$N = Ce^{kt}$, $C = 120$, $k = \frac{1}{4}\ln(0.3)$, so $N = 120(0.3)^{t/4}$.
When $t = 8$, $N = 10.8$ g.

4. $dP/dt = kt/P$, $P(0) = 10$ (thousands), $P(10) = 20$.
$P^2 = kt^2 + C$, $C = 100$, $k = 3$, so $P^2 = 3t^2 + 100$.
$P = 52$ when $t^2 = 868$, so $t \approx 29$ years (the year 2009).

5. $dP/dt = kP/t$, $P(1) = 800$ (thousands), $P(9) = 848$.
$P = Ct^k$, $C = 800$, $k = \ln(1.06)/\ln 9 \approx 0.02652$.
$P(18) \approx 863.73$ (about 864,000).

6. $dS/dt = k/\sqrt{t}$, $S(0) = 40$ (millions of \$), $S(1) = 50$.
$S = 2k\sqrt{t} + C$, $C = 40$, $k = 5$, so $S = 10\sqrt{t} + 40$.
$S(4) = 60$ (\$60 million).

7. $dS/dt = kS^2t^2$, $S(0) = 100$ (millions of \$), $S(3) = 120$.
$-1/S = \frac{1}{3}kt^3 + C$, $C = -1/100$, $k = 1/5400$, so $S = 16{,}200/(162 - t^3)$.
When $S = 165$, $t^3 = 702/11$, so $t \approx 4$ years.

8. $dN/dt = kNt$, $N(0) = 400$, $N(2) = 1200$.
$N = Ce^{kt^2/2}$, $C = 400$, $k = \frac{1}{2}\ln 3$, so $N = 400 \cdot 3^{t^2/4}$.
$N(4) = 32{,}400$ units.

9. $dV/dt = kV$, $V(0) = 36$ (thousands of \$), $V(2) = 18$.
$V = Ce^{kt}$, $C = 36$, $k = \frac{1}{2}\ln(0.5)$, so $V = 36(0.5)^{t/2}$.
$V(4) = 9$ (\$9000).

10. $dV/dt = k\sqrt{V}$, $V(0) = 1600$ (\$), $V(2) = 900$.
$\sqrt{V} = \frac{1}{2}kt + C$, $C = 40$, $k = -10$, so $\sqrt{V} = 40 - 5t$.
When $V = 625$, $t = 3$ years.

11. Letting $N(t)$ be the number of people who have heard the rumor after t weeks, we have $dN/dt = k(10{,}000 - N)$, $N(0) = 25$, $N(3) = 6675$.
$N = 10{,}000 - Ce^{-kt}$, $C = 9975$, $k = -\frac{1}{3}\ln\frac{1}{3} = \frac{1}{3}\ln 3$, so $N = 10{,}000 - 9975(\frac{1}{3})^{t/3} = 10{,}000 - 9975 \cdot 3^{-t/3}$.
$N(6) = 8891\frac{2}{3} \approx 8892$ people.

12. $dN/dt = kNt$, $N(0) = 100$, $N(1) = 250$.
$N = Ce^{kt^2/2}$, $C = 100$, $k = 2\ln(2.5)$, so $N = 100(2.5)^{t^2}$.
When $N = 500$, $t^2 = \ln 5/\ln(2.5)$, so $t \approx 1.33$ years.

13. $dV/dt = kV^2$, $V(0) = 2500$ (\$), $V(2) = 1500$.
$-1/V = kt + C$, $C = -1/2500$, $k = -1/7500$, so $V = 7500/(t + 3)$.
$V(3) = 1250$ (\$1250).

14. $dP/dt = kt/P$, $P(0) = 10$ (thousands), $P(10) = 20$.
$P^2 = kt^2 + C$, $C = 100$, $k = 3$, so $P^2 = 3t^2 + 100$.
$P = 41$ when $t^2 = 527$, so $t \approx 23$ years.

15. $dN/dt = kt/N$, $N(0) = 1$, $N(5) = 100$.
$N^2 = kt^2 + C$, $C = 1$, $k = 399.96$, so $N^2 = 399.96t^2 + 1$.
When $t = 25$, $N^2 = 249{,}976$, so $N \approx 500$ people.

16. $dN/dt = kt^2$, $N(0) = 13$ (hundreds), $N(1) = 21$.
$N = \frac{1}{3}kt^3 + C$, $C = 13$, $k = 24$, so $N = 8t^3 + 13$.
$N = 229$ when $t^3 = 27$, so $t = 3$ years.

17. $dN/dt = kNt^2$, $N(0) = 8$, $N(1) = 16$.
$N = Ce^{kt^3/3}$, $C = 8$, $k = 3\ln 2$, so $N = 8 \cdot 2^{t^3}$.
$N(2) = 2048$ units.

18. $dV/dt = kV/\sqrt{t}$, $V(0) = 2500$ (\$), $V(4) = 1600$.
$V = Ce^{2k\sqrt{t}}$, $C = 2500$, $k = \frac{1}{4}\ln(0.64)$, so
$V = 2500(0.64)^{\sqrt{t}/2}$.
When $V = 1280$, $\sqrt{t} = 2\ln(0.512)/\ln(0.64) = 3$, so $t = 9$ years.

19. $dP/dt = kPt^2$, $P(0) = 100$ (thousands), $P(3) = 300$.
$P = Ce^{kt^3/3}$, $C = 100$, $k = \frac{1}{9}\ln 3$, so $P = 100 \cdot 3^{t^3/27}$.
When $N = 1200$, $\frac{1}{27}t^3 = \ln 12/\ln 3$, so $t \approx 3.9$ years.

20. $dN/dt = kN/t^2$, $N \to 5$ (thousands) as $t \to \infty$, $N(1) = 1$.
$N = Ce^{-k/t}$, $C = 5$ because $N \to C$ as $t \to \infty$, $k = -\ln(\frac{1}{5}) = \ln 5$, so $N = 5(\frac{1}{5})^{1/t} = 5^{1-1/t}$.
$N(5) \approx 3.624$ (3624 units).