1. Determine whether the improper integral is convergent or divergent. If the integral converges, give its exact value. *Use correct notation throughout.*

(a) $\displaystyle\int_{\pi/2}^{\pi} \cot x\,dx$ (b) $\displaystyle\int_0^4 \frac{dx}{(x-2)^4}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+16}$ (d) $\displaystyle\int_2^{\infty} \frac{dx}{2x^2+x-1}$

(e) $\displaystyle\int_0^1 \frac{e^x}{1-e^x}\,dx$ (f) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1-\tan x}}\,dx$

(g) $\displaystyle\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ (h) $\displaystyle\int_0^4 \frac{dx}{x^2-2x-3}$

(i) $\displaystyle\int_0^{\pi} \frac{\cos x}{\sqrt{\sin x}}\,dx$ (j) $\displaystyle\int_6^{\infty} \frac{dx}{x\sqrt{x^2-9}}$

2. Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = 4x - x^2$. Set up, but do not evaluate the integrals for the area of $\mathcal{R}$,

3. Sketch the region $\mathcal{R}$ bounded by the parabola $x = y^2$ and the line $y = x - 2$. Find the area of $\mathcal{R}$.

4. Sketch the region $\mathcal{R}$ in the first quadrant bounded by the parabola $y = 3 + 2x - x^2$ and the lines $y = 4x$ and $y = \frac{3}{2}x$. Then find the area of $\mathcal{R}$.

5. Sketch the region $\mathcal{R}$ bounded by the curves $y = 2\sin x$ and $y = \tan x$ between $x = 0$ and $x = \frac{1}{3}\pi$. Then find the area of $\mathcal{R}$.

6. Determine whether the sequence is convergent or divergent. Justify your answer: if it converges, find the limit; otherwise, indicate why it diverges.

(a) $\left\{ n\ln\left(\frac{1}{n}\right) \right\}$ (b) $\left\{ \frac{e^n - 4n}{n^2+1} \right\}$

(c) $\left\{ \frac{3n^2(n-1)!}{5(n+1)!} \right\}$ (d) $\arcsin\left(\frac{-n}{(\sqrt{n}+3)^2} \right)$

(e) $\left\{ (-1)^n \frac{n+1}{2n-1} \right\}$ (f) $\left\{ \frac{\cos n}{3n+1} \right\}$

Answers

1. (a) Diverges to $-\infty$. The integral equals

$$\lim_{t\to\pi^-} \ln|\sin x| \Big]_{\pi/2}^{t} = \lim_{t\to\pi^-} \ln|\sin t| = -\infty$$

(b) Diverges (to ∞). Because of the vertical asymptote at $x = 2$, the integral equals

$$\int_0^2 \frac{dx}{(x-2)^4} + \int_2^4 \frac{dx}{(x-2)^4}$$

Using the anti-derivative $-\frac{1}{3}(x-2)^{-3}$, you can check that the first integral diverges (to ∞), and so the original integral also diverges. There is no need to consider the second integral (which also happens to diverge to ∞).

(c) Converges to $\frac{1}{4}\pi$. The indefinite integral is $\frac{1}{4}\arctan(\frac{1}{4}x) + C$.

(d) Converges to $\frac{1}{3}\ln 2$. Using partial fractions, the indefinite integral is

$$\tfrac{1}{3}\ln|2x-1| - \tfrac{1}{3}\ln|x+1| + C = \tfrac{1}{3}\ln\left|\frac{2x-1}{x+1}\right| + C$$

(e) Diverges to $-\infty$. The integral equals

$$\lim_{t\to 0^+} -\ln|1-e^x| \Big]_t^1 = \lim_{t\to 0^+} \left(-\ln|1-e| + \ln|1-e^t|\right) = -\infty$$

(f) Converges to 2. The integral equals

$$\lim_{t\to(\pi/4)^-} -2\sqrt{1-\tan x}\Big]_0^t = \lim_{t\to(\pi/4)^-} -2\sqrt{1-\tan t} + 2 = 2$$

(g) Converges to 2. Because of the vertical asymptote at $x = 0$, we have to split this up into two integrals:

$$\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx + \int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$
$$= \lim_{t\to 0^+} -2e^{-\sqrt{x}}\Big]_t^1 + \lim_{t\to\infty} -2e^{-\sqrt{x}}\Big]_1^t = 2 - 2/e + 2/e$$

(h) Diverges. Because of the vertical asymptote at $x = 3$, we have to split this up into two integrals:

$$\int_0^3 \frac{dx}{x^2-2x-3} + \int_3^4 \frac{dx}{x^2-2x-3}$$

Using partial fractions, the first integral equals

$$\lim_{t\to 3^-} \tfrac{1}{4}\left(\ln|x-3| - \ln|x+1|\right)\Big]_0^t = -\infty$$

(i) Converges to 0. Because of the vertical asymptotes at $x = 0$ and $x = \pi$, we have to split this up into two integrals:

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx + \int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}}\,dx$$
$$= \lim_{t\to 0^+} 2\sqrt{\sin x}\Big]_t^{\pi/2} + \lim_{t\to\pi^-} 2\sqrt{\sin x}\Big]_{\pi/2}^t = 2 - 2$$

(j) Converges to $\frac{1}{18}\pi$. The integral equals

$$\lim_{t\to\infty} \tfrac{1}{3}\operatorname{arcsec}(\tfrac{1}{3}x)\Big]_6^t = \tfrac{1}{3}(\tfrac{1}{2}\pi - \tfrac{1}{3}\pi)$$

2. $\int_0^2 (4x - 2x^2)\, dx$

3. $\int_{-1}^2 (y + 2 - y^2)\, dy = \frac{9}{2}$

4. $\int_0^1 \frac{5}{2}x\, dx + \int_1^2 (3 + \frac{1}{2}x - x^2)\, dx = \frac{8}{3}$

5. $1 - \ln 2$

6. (a) As $n \to \infty$, $1/n \to 0^+$, so $\ln(1/n) \to -\infty$ (recall that the graph of $y = \ln x$ has a vertical asymptote at $x = 0$). It follows that $n\ \ln(1/n) \longrightarrow -\infty$, i.e., the sequence diverges. Or, we can see this by simply noting that

$$n \ln(1/n) = n(\ln 1 - \ln n) = -n \ln n$$

(b) Diverges (to ∞). By l'Hôpital's rule, the limit equals

$$\lim_{x\to\infty} \frac{e^x - 4x}{x^2 + 1} = \lim_{x\to\infty} \frac{e^x - 4}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

(or focus on the dominant terms e^x in the numerator and x^2 in the denominator).

(c) Converges to $\frac{3}{5}$ because

$$\frac{3n^2(n-1)!}{5(n+1)!} = \frac{3n^2}{5(n+1)n} = \frac{3}{5(1+1/n)} \longrightarrow \frac{3}{5} \quad \text{as } n \to \infty$$

(d) Converges to $\arcsin(-1) = -\frac{\pi}{2}$.

(e) Diverges by oscillation because

$$\frac{n+1}{2n-1} \longrightarrow \frac{1}{2} \neq 0 \quad \text{as } n \to \infty$$

(f) Converges to 0 because

$$0 \leqslant \left| \frac{\cos n}{3n+1} \right| \leqslant \frac{1}{3n+1} \longrightarrow 0 \quad \text{as } n \to \infty$$