Bases and Basis Selection

Definition: Let S be a subspace of $\mathbb{R}^n$. The set $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$ of vectors in S is called a **basis** for S if both of the following are true:
1) the vectors $\vec{u_1}, \vec{u_2}, ..., \vec{u_k}$ are linearly independent and
2) $S = span\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$
Each of the individual vectors is called a **basis vector**.

Basically, a **basis** of S is the smallest set of vectors needed to span S.

Theorem: BASES ARE NOT UNIQUE! However, all bases for S will contain the same number of vectors.
All bases of a line through the origin will contain 1 vector.
All bases of a plane through the origin or $\mathbb{R}^2$ will contain 2 vectors.
All bases of $\mathbb{R}^3$ will contain 3 vectors.
All bases of $\mathbb{R}^n$ will contain n vectors.

Theorem: Let S be a subspace of $\mathbb{R}^n$. Then every basis of S has the same number of vectors, and the number of vectors in a basis for S is called the **dimension** of S.

The above can be easily summarized in the following chart.

Subspace	Dimension
$\{\vec{0}\}$	0
line through O	1
plane through O	2
$\mathbb{R}^n$	n

Theorem: If $U = \begin{bmatrix} \vec{u_1} & \vec{u_2} & ... & \vec{u_n} \end{bmatrix}$ and $\det(U) \neq 0$, then $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_n}\}$ is linearly independent and spans $\mathbb{R}^n$ and is therefore a basis for $\mathbb{R}^n$.
Let S be a k-dimensional subspace (need k basis vectors to span all of S) of $\mathbb{R}^n$. If a set of k vectors in S either spans S or is linearly independent, then it is both (that is, one implies the other) and is therefore a basis for S.

Example: Which of the following sets of vectors form a basis for $\mathbb{R}^3$?

a) $\{(1,3,5), (-2,6,1)\}$ does not form a basis for $\mathbb{R}^3$ because they do not span $\mathbb{R}^3$. In this case, they span a plane in $\mathbb{R}^3$ because they are linearly independent (not scalar multiples).

b) $\{(1,3,5), (-2,6,1), (0,4,2), (-3,0,1)\}$ does not form a basis for $\mathbb{R}^3$ because the vectors are linearly dependent ($k > n$: the number of vectors $>$ the number of components)

c) $\{(1,3,5), (-2,6,1), (0,4,2)\}$ has the correct number of vectors. To determine if the set forms a basis, we must just check to see if the vectors are linearly independent. We can do this by evaluating the determinant.

$$\begin{vmatrix} 1 & -2 & 0 \\ 3 & 6 & 2 \\ 5 & 1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & -2 & 0 \\ -7 & 4 & 0 \\ 5 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix} = 2(4-14) = -20 \neq 0.$$

The determinant does not equal 0, so the vectors are linearly independent, and this set of vectors is therefore a basis for $\mathbb{R}^3$.

d) $\{(1,3,5),(-2,6,1),(7,-3,13)\}$ has the correct number of vectors. To determine if the set forms a basis, we must just check to see if the vectors are linearly independent. We can do this by evaluating the determinant.

$$\begin{vmatrix} 1 & -2 & 7 \\ 3 & 6 & -3 \\ 5 & 1 & 13 \end{vmatrix} \to \begin{vmatrix} 1 & -2 & 7 \\ 0 & 12 & -24 \\ 0 & 11 & -22 \end{vmatrix} = 1 \begin{vmatrix} 12 & -24 \\ 11 & -22 \end{vmatrix} = ((12)(-22) - (-24)(11)) = 0.$$

The determinant equals 0, so the vectors are linearly dependent, and this set of vectors is not a basis for $\mathbb{R}^3$.

Example: Find a basis for $S = \{(x,y,z) \in \mathbb{R}^3 | x + y - z = 0\}$

Since we know that a basis is a set of linearly independent vectors that span the space, we need only find the two (because it's a plane) linearly independent vectors in the span of this space.

Find the parametric solutions of $\begin{bmatrix} 1 & 1 & -1 | 0 \end{bmatrix}$: $\begin{cases} x = -t + s \\ y = t \\ z = s \end{cases}$, so

$S = \text{span}\{(-1,1,0),(1,0,1)\}$ and therefore $\{(-1,1,0),(1,0,1)\}$ is a basis for S.