Intro to Vectors

A **vector** is an ordered set of numbers that represents length (magnitude) and direction.

Notation: A vector is typically denoted either in bold, **v**, or with an arrow above it $\vec{v}$. We typically use the same letter for the different components of a vector. So a vector in 3-space with 3 components might be written $\vec{v} = \langle v_1, v_2, v_3 \rangle$ usually, or sometimes $\vec{v} = (v_1, v_2, v_3)$ when there is no ambiguity. We also write them as column matrices (i.e. matrices that have only one column).

When considering a vector $\vec{u} = \langle u_1, u_2 \rangle$ in $\mathbb{R}^2$, we must understand that u_1 represents the displacement in the x-direction and u_2 represents the displacement in the y-direction - it does NOT represent a location; just a length and a direction.

Equality of Vectors: Two vectors are said to be **equal** if their components are equal; that is, if they have the same direction and magnitude.

If we want for a vector to be located in a specific place, it is necessary to give the vector an **initial point**, which indicates where the vector will start. In this case, the point where it ends will be called the **terminal point**.

A vector between two points: If $P_1(x_1, y_1)$ is the initial point and $P_2 = (x_2, y_2)$ is the terminal point for the vector $\vec{u}$, then $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$

If a vector is placed so that its initial point is the origin, O, with a terminal point of P, then it is called a **position vector**. Unless we are specifically given initial points or terminal points, we will think of any vector in $\mathbb{R}^n$ is a position vector.

Vector Operations

Addition/Subtraction of Vectors: If $\vec{u} = \langle u_1, u_2, ...u_n \rangle$ and $\vec{v} = \langle v_1, v_2, ..., v_n \rangle$ are vectors then $\vec{u} \pm \vec{v} = \langle u_1 \pm v_1, u_2 \pm v_2, ..., u_n \pm v_n \rangle$
This means to first do the length and direction of $\vec{u}$ and then do the length and direction of $\vec{v}$ (or vice versa).

Scalar Multiplication of Vectors: If $\vec{v} = \langle v_1, v_2, ...v_n \rangle$ is a vector and k is a scalar, then $k\vec{v} = \langle kv_1, kv_2, ..., kv_n \rangle$

Scalar multiplication of a vector does the following:
1) Stretches it (makes it longer) if $k > 1$
2) Shrinks it (makes it shorter) if $0 < k < 1$
3) Changes its direction if $k < 0$

Definition: Two vectors are said to be **parallel** if they point in the same or opposite directions (if the angle between them is 0)

Example: The vector $\vec{u} = \langle 3, 5, -2, 4 \rangle$ is parallel to the vector $\vec{v} = \langle -6, -10, 4, -8 \rangle$ because they point in opposite directions. This can be seen because $\vec{v}$ is a scalar multiple of $\vec{u}$.

Properties of Vectors

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let a and b be scalars. Then:
a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
c) $a(b\vec{u}) = (ab)\vec{u}$
d) $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
e) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
f) $\vec{u} + \vec{0} = \vec{u}$
g) $1(\vec{u}) = \vec{u}$
h) $a(\vec{0}) = \vec{0}$, and $0(\vec{u}) = \vec{0}$
i) $\vec{u} + (-\vec{u}) = \vec{0}$
j) If $a\vec{u} = \vec{0}$, then either $a = 0$ or $\vec{u} = 0$

The Magnitude (or Norm) of a Vector

The **magnitude** of a vector $\vec{v} = \langle v_1, v_2, ..., v_n \rangle$ is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$

Definition: A vector of norm 1 is called a **unit vector**.

Given a non-zero vector $\vec{v}$ in $\mathbb{R}^n$, $\vec{u} = \dfrac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector in the same direction as $\vec{v}$.

Theorem: If $\vec{v}$ is any vector in $\mathbb{R}^n$, then
a) $\|\vec{v}\| \geq 0$
b) $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = 0$
c) $\|k\vec{v}\| = |k| \, \|\vec{v}\|$

The Dot Product
If $\vec{u} = \langle u_1, u_2, ..., u_n \rangle$ and $\vec{v} = \langle v_1, v_2, ..., v_n \rangle$ are vectors in $\mathbb{R}^n$ then the **dot product** of $\vec{u}$ and $\vec{v}$ is given by $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + ... + u_n v_n$.

Example: Find the dot product of $\vec{u} = \langle 1, 3, 2, 0 \rangle$ and $\vec{v} = \langle 6, -3, 1, 2 \rangle$.

$\vec{u} \cdot \vec{v} = (1)(6) + (3)(-3) + (2)(1) + (0)(2) = 6 - 9 + 2 = -1$

Theorem: Two non-zero vectors $\vec{u}$ and $\vec{v}$ are said to be **orthogonal** (or perpendicular) if $\vec{u} \cdot \vec{v} = 0$. We write $\vec{u} \perp \vec{v}$

Definition: A non-empty set of vectors in $\mathbb{R}^n$ is said to be an **orthogonal set** if ALL pairs of vectors in the set are orthogonal.

Example: Do the vectors $\vec{u} = \langle 1, 2, 3 \rangle, \vec{v} = \langle -4, 5, -2 \rangle$, and $\vec{w} = \langle 3, 6, -5 \rangle$ form an orthogonal set?

Check each pair separately:
$\vec{u} \cdot \vec{v} = (1)(-4) + (2)(5) + (3)(-2) = -4 + 10 - 6 = 0$, so $\vec{u} \perp \vec{v}$
$\vec{u} \cdot \vec{w} = (1)(3) + (2)(6) + (3)(-5) = 3 + 12 - 15 = 0$, so $\vec{u} \perp \vec{w}$

$\vec{v} \cdot \vec{w} = (-4)(3) + (5)(6) + (-2)(-5) = -12 + 30 + 10 = 28 \neq 0$, so $\vec{v}$ is not perpendicular to $\vec{w}$. Because $\vec{v}$ and $\vec{w}$ are not orthogonal, these vectors do not form an orthogonal set.

The Cross Product

If If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in $\mathbb{R}^3$ then the **cross product** of $\vec{u}$ and $\vec{v}$ is

given by $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{i}} & u_1 & v_1 \\ \hat{\mathbf{j}} & u_2 & v_2 \\ \hat{\mathbf{k}} & u_3 & v_3 \end{vmatrix}$, where $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle, \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle, \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$.

or in other words, $\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \right\rangle$

Example: Find the cross product of $\vec{u} = \langle 1, 2, 3 \rangle$ and $\vec{v} = \langle -2, 1, 4 \rangle$

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{\mathbf{i}} & 1 & -2 \\ \hat{\mathbf{j}} & 2 & 1 \\ \hat{\mathbf{k}} & 3 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \hat{\mathbf{k}} \\ &= 5\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 5\hat{\mathbf{k}} \\ &= \langle 5, -10, 5 \rangle \end{aligned}$$

Theorem: If $\vec{u}, \vec{v}$ are two non-parallel non-zero vectors in $\mathbb{R}^3$, then $\vec{w} = \vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Note: Otherwise, their cross product is simply $\vec{0}$. Why is that?