The Leontief Open Model

The **Leontief Open Model** is a simplified economic model for an economy in which consumption equals production, or input equals output. **Internal Consumption** (or internal demand) is defined to be the amount of production consumed within the industries themselves, whereas **External Demand** is the amount that is used outside of the industries. This model allows us to determine how much production is needed in the various industries to meet the total demand.

In order to determine the production level, we need to properly set up a consumption matrix C. Each entry c_{ij} corresponds to the number of units of output from industry i required (or consumed) to produce one unit of output of industry j. For our purposes, our unit will always be the almighty dollar.

We also need to construct the external demand vector D which gives the external demand for each industry.

Let C be the **consumption matrix**,
let D be the **demand vector**,
and let X be the **production vector**.
Then CX is the **internal consumption**.

Starting with Production = Total Demand (Internal and External), we solve for X.
$X = CX + D$
$X - CX = D$
$(I - C)X = D$
Now multiply both sides on the left by $(I - C)^{-1}$, giving
$\boxed{X = (I - C)^{-1}D}$

Note: This of course only works if $I - C$ is invertible.

Definition: An economy is productive if it can meet any external demand. In other words, for any D, there is an X such that $X - CX = D$ with $X \geq 0$ (i.e. X contains only non-negative entries).

Consequently an economy is **productive** if $(I - C)$ is invertible and $(I - C)^{-1} \geq 0$

Definition: An industry is **profitable** if the corresponding column in C has sum less than 1.

Theorem 1: If a consumption matrix C for an economy is such that either
(a) For every column in C, the sum of the entries in the column is less than 1 OR
(b) For every row in C, the sum of the entries in the row is less than 1,
then the economy is productive.

Example: Let an economy contain three industries: electric, oil, and pipeline. For each \$1 of electricity generated, let there be charges of 20¢ electricity, 40¢ for oil, and 10¢ in pipeline. For each \$1 of oil producted, suppose there are charges of 10¢ in electricity and 40¢ for oil. Finally, for each \$1 in pipeline, suppose that 30¢ is spent on electricity, while 20¢ is spent on oil. There is an external demand for \$4200 worth of electricity, \$8400 in oil, and \$12600 in pipeline. How much should each industry produce to meet the demand? What is the internal consumption?

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0 & 0 \end{bmatrix}, \text{ so } I - C = \begin{bmatrix} 0.8 & -0.1 & -0.3 \\ -0.4 & 0.6 & -0.2 \\ -0.1 & 0 & 1 \end{bmatrix} \text{ and } (I-C)^{-1} = \begin{bmatrix} \frac{10}{7} & \frac{5}{21} & \frac{10}{21} \\ 1 & \frac{11}{6} & \frac{2}{3} \\ \frac{1}{7} & \frac{1}{42} & \frac{22}{21} \end{bmatrix}$$

The (external) demand vector is $D = \begin{bmatrix} 4200 \\ 8400 \\ 12600 \end{bmatrix}$, thus $X = (I-C)^{-1}D = \begin{bmatrix} 14000 \\ 28000 \\ 14000 \end{bmatrix}$

This means \$14000 of electricity, \$28000 of oil and \$14000 of pipeline must be produced to meet the demand.

The internal consumption is easy to compute. It's $CX = X - D$, so equals $\begin{bmatrix} 9800 \\ 19600 \\ 1400 \end{bmatrix}$

Note that since $(I-C)^{-1}$ is non-negative, the economy is productive and no matter what the demand is, there is a production vector that can meet it. In fact, we could figure that out before doing any calculations. All three industries are profitable since each column sum is less than 1. So Theorem 1 applies to this example.

Example: Consider a simple economy with two industries, coal and steel. Each consumes some of its own production and some from the other industry.

The coal industry consumes 20% of its own production as well as an amount of steel which contributes to 40% of its own output. The steel industry consumes 10% of its own production as well as an amount of coal which contributes to 30% of its own output. The external demand from other sources is \$24 million worth of coal and \$30 million worth of steel. What is the total value of the output required of the coal and steel industries in order to meet all demand?

Example: An economy consists of two industries, Pink and Turquoise. Production of one dollar's worth of Pink requires an input of \$0.70 of Pink and \$0.20 of Turquoise. Production of one dollar's worth of Turquoise requires an input of \$0.50 of Pink and \$0.60 of Turquoise.

a) Find the total production to satisfy a demand of \$14 million of Pink and \$10 million of Turquoise.
b) What amount of Pink and Turquoise are consumed internally?
c) Which, if any, of the two industries are profitable? Explain.
d) Is the economy productive? Explain.