<h1 style="text-align:center">Linear Independence and Linear Dependence</h1>

Definition: The set of k vectors $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$ in $\mathbb{R}^n$ is said to be **linearly dependent** if one or more of the vectors can be written as a linear combination of the other vectors.
The set of vectors is called **linearly independent** if it is not linearly dependent.
A set $\{\vec{u}\}$ consisting of a single vector is called linearly dependent iff $\vec{u}$ is the zero vector.

In other words, the set of k vectors $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$ in $\mathbb{R}^n$ is said to be **linearly dependent** if there exist scalars $c_1, c_2, ...c_k$, with at least one of the scalars not 0, such that

$$c_1\vec{u_1} + c_2\vec{u_2} + ... + c_k\vec{u_k} = 0$$

This equation is called a **dependency equation**.

They are **linearly independent** iff the only scalars that satisfy $c_1\vec{u_1} + c_2\vec{u_2} + ... + c_k\vec{u_k} = 0$ are ALL 0!!

Example: The set of vectors $\{(1,2), (3,0), (-1,4)\}$ is linearly dependent because
$$-2(1,2) + (3,0) + (-1,4) = 0$$
That is, we were able to form a dependency equation in which at least one the the constants wasn't 0.

Tips for Determining Linear Dependency Let $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$ be a set of k vectors in $\mathbb{R}^n$. Then:

1. If $k > n$, then the vectors are linearly dependent.
2. If $\vec{0} \in \{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$, then the vectors are linearly dependent.
3. If $k = 2$ and the vectors are proportional (scalar multiples), then they are linearly dependent. If they are not scalar multiples, they are linearly independent.
4. If $k = n$ and $\det(U) = 0$, then the vectors are linearly dependent. If $\det(U) \neq 0$, then the vectors are linearly independent.
   (U is the matrix formed with the vectors in the set as the columns)

Steps for Determining if Vectors are Linearly Independent (L.I.) or Linearly Dependent (L.D.)
To determine if a set of vectors $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_k}\}$ in $\mathbb{R}^n$ are L.I. or L.D.,

Step 1: Determine if the Tips for Determining Dependency apply.
Step 2: If they don't, create the matrix U whose columns are $\vec{u_1}, \vec{u_2}, ..., \vec{u_k}$.
Step 3: Solve the system $UX = \vec{0}$ where $X = (c_1, c_2, ..., c_k)$. This can most easily done by row reducing the augmented matrix with 0 on the right of the augmentation.
Step 4: If the system has only the trivial solution $c_1 = 0, c_2 = 0, ..., c_k = 0$, then the vectors are **linearly independent**. (In this case, every column has a pivot).
If the system has infinitely many (non-trivial) solutions, then the vectors are **linearly dependent**. (In this case, there will be at least one column with no pivot.)

Theorem: Given vectors $\vec{u_1}, \vec{u_2}, ..., \vec{u_k}$ in $\mathbb{R}^n$, if $k = n$ (that is, the number of vectors is the same as the number of elements in the vector) and if the vectors are linearly independent (the determinant of matrix formed from the column vectors is not 0), then the vectors span $\mathbb{R}^n$.

Important Note: If the vectors do not span $\mathbb{R}^n$, we can easily determine if they span a line or a plane by looking at the number of linearly independent vectors. The number of linearly independent vectors will be the number of pivots in the row reduced form.

Example: Are the vectors $\vec{u_1} = (3, 4, 1), \vec{u_2} = (-1, 3, 2)$, and $u_3 = (4, 1, -1)$ linearly dependent or independent? Do they span a plane, a line, or $\mathbb{R}^3$?

Step 1: In this case, $k = n$, so we can set up U and find the determinant.
$$|U| = \begin{vmatrix} 3 & -1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} - 4 \begin{vmatrix} -1 & 4 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} = (3)(-5) - 4(-7) + (1)(-13) = 0$$
We can see that $\det(U) = 0$, so we know that these vectors are **linearly dependent**. Because the three vectors are linearly dependent, they do not span $\mathbb{R}^3$. To determine their span, we must continue the steps.

Step 2: $U = \begin{bmatrix} 3 & -1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & -1 \end{bmatrix}$

Step 3: $\xrightarrow[\substack{R_2 - 4R_3}]{R_1 - 3R_3} \begin{bmatrix} 0 & -7 & 7 \\ 0 & -5 & 5 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & -5 & 5 \end{bmatrix} \xrightarrow{R_2(1/-7)}$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{bmatrix} \xrightarrow[\substack{R_3 + 5R_2}]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Step 4: This system clearly has infinite solutions, so we can conclude the same thing as in step 1: the vectors are linearly dependent.

We see, though, that there are 2 pivots in the reduced matrix. From this we conclude that there are two linearly independent vectors (see above Important Note). We know that **the span of two linearly independent vectors in $\mathbb{R}^3$ is a plane** (see notes on Span).