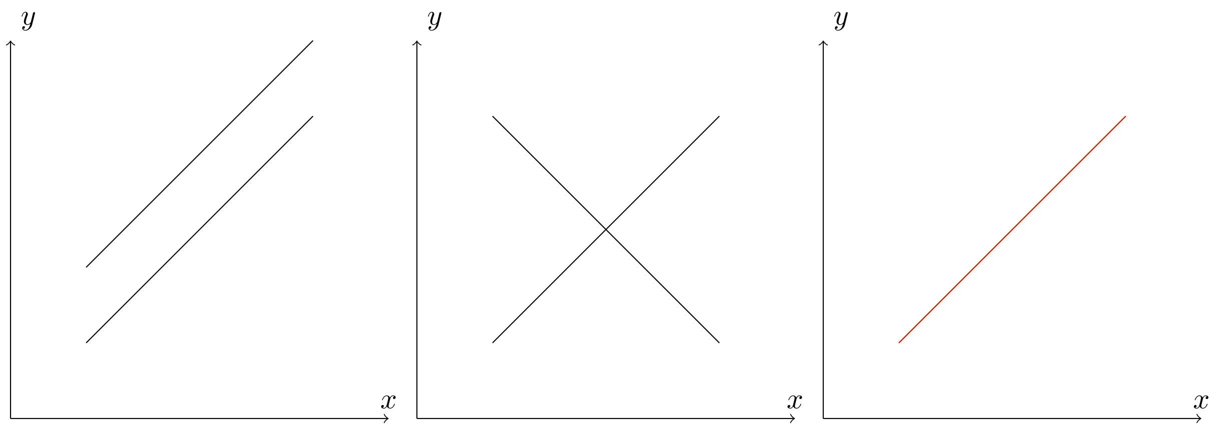
LINEAR SYSTEMS

Theorem: Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Consider the following graphs:

No Solutions One Solution Infinite Solutions (the same line)



Definition: An **augmented matrix** is a matrix formed from the coefficients in a linear system. The rightmost column contains the constant terms.

Example: Write the augmented matrix for the following system of equations.

$$\begin{cases} 7x + 6y - 2z = 4 \\ 2x + 3y + 4z = 8 \\ 12x + \qquad 3z = 7 \end{cases}$$

Solution: The augmented matrix for the previous system is the following: $\left[\begin{array}{ccc|c} 7 & 6 & -2 & 4 \\ 2 & 3 & 4 & 8 \\ 12 & 0 & 3 & 7 \end{array}\right]$

Note: It is also easy to go from the augmented matrix back to the system of equations.

Augmented matrices are useful because they can be easily reduced by a sequence of elementary row operations to reveal the solution(s) to the system of equations.

Elementary Row Operations

1. Multiply a row through by a nonzero constant.
2. Interchange/swap two rows.
3. Change a row by adding a multiple of another row to it.

Our goal will be to reduce these matrices to **reduced row echelon form**, which is characterized by the following properties:

Reduced Row Echelon Form (RREF)

1. If a row does not consist entirely of 0s, then the first nonzero number in the row is a 1. This is the **pivot** position, and it is called the **leading 1**.
2. If there are any rows that consist entirely of 0s, they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of 0s, the leading 1 in the lower row occurs farther to the right than the leading one in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

Examples of matrices in RREF:

2 unknowns (x and y) 3 unknowns ($x, y,$ and z) 4 unknowns ($x, y, z,$ and w)

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & * \\ 0 & 1 & 3 & * \\ 0 & 0 & 0 & * \end{array}\right] \quad \left[\begin{array}{cccc|c} 0 & 1 & -2 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

To reduce a matrix to RREF, we use the procedure below (known as Gauss-Jordan elimination):

Moving from left to right, transform one column at a time
For each column:

1. **Select a pivot (if possible)**
   The pivot must be non-zero and lie in a row below all previous pivot rows.
   Pivots of 1 are desirable, but so are pivots that are factors of all other elements in their column. If neither of these is an option, you can create a pivot of 1 by
   i. multiplying the pivot row by the reciprocal of the pivot.
   ii. adding or subtracting a multiple of another row.
   If necessary, interchange rows so that the pivot row is as high as possible (but still below all previous pivot rows).
2. **Create 0s**
   Add (or subtract) multiples of the pivot row to each row above and beneath it. Choose the multiples so that all non-pivot entries in the column become 0.

When the procedure is completed, your matrix will be in reduced row echelon form (RREF).

REMARKS: The procedure itself is flexible, so once you understand it, feel free to play around with it. Also, note that we can use the procedure itself to check whether a matrix is in RREF. If we apply Gauss-Jordan elimination to a matrix already in RREF, then it remains unchanged.

UNIQUE SOLUTION: A system with a unique solution has an augmented matrix that when reduced to RREF has a leading 1 (pivot) in every column in the coefficient matrix, but no pivot in the last column of the augmented matrix.

Example 1: Solve the following linear system

$$\begin{cases} 3x - 7y + 11z = 4 \\ x - 2y + 3z = 1 \\ -2x + 8y - 16z = 4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -7 & 11 & 4 \\ 1 & -2 & 3 & 1 \\ -2 & 8 & -16 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 3 & -7 & 11 & 4 \\ -2 & 8 & -16 & 4 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & -1 & 2 & 1 \\ -2 & 8 & -16 & 4 \end{array}\right] \xrightarrow{R_3 + 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 4 & -10 & 6 \end{array}\right] \xrightarrow[R_3 - 4R_2]{R_2/-1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -2 & 10 \end{array}\right] \xrightarrow[R_3/-2]{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{R_2 + 2R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & -5 \end{array}\right]$$ We conclude that $x = -6, y = -11$, and $z = -5$.

NO SOLUTIONS: A system with no solutions is called **inconsistent**. In general, to show that a system is inconsistent, reduce as usual. Keep reducing until the matrix produces a contradiction, in the form of

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & \dots & 0 & k \end{array}\right] \text{ where } k \text{ is any non-zero constant}$$

Example 2: Solve the following linear system

$$\begin{cases} 3x - 3y - 6z = -3 \\ 2x - 2y - 4z = 10 \\ -2x + 3y + z = 7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -3 & -6 & -3 \\ 2 & -2 & -4 & 10 \\ -2 & 3 & 1 & 7 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 2 & -2 & -4 & 10 \\ -2 & 3 & 1 & 7 \end{array}\right] \xrightarrow[R_3 + 2R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 12 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

Recalling that each row represents an equation, R_2 now gives the equation $0 = 12$. This equation can never be true. Thus we obtain a contradiction and the system has no solutions.

INFINITELY MANY SOLUTIONS: If a system has infinitely many solutions, then at least one column in the coefficient matrix does not have a pivot. A set of **parametric equations** from which all solutions can be obtained by assigning numerical values to the parameters is called the **general solution**.

Note: If the reduced echelon form of a consistent system has a column without a leading one (or pivot), then the system has infinitely many solutions. Each variable corresponding to a column with no pivot is called a **free variable**.

Example 3:

$$\begin{cases} 3x - 3y - 6z = -3 \\ 2x - 2y - 4z = -2 \\ -2x + 3y + z = 7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -3 & -6 & -3 \\ 2 & -2 & -4 & -2 \\ -2 & 3 & 1 & 7 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 2 & -2 & -4 & -2 \\ -2 & 3 & 1 & 7 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Once the matrix is reduced, we can see that z is a **free variable**, as the third column has no pivot. So let $z = t$, and thus $x = 4 + 5t, y = 5 + 3t, z = t$.

Definition: The **row rank** of a matrix is the number of leading numbers when put into echelon form.

The rank of the matrix from Example 1 is 3. The rank of the matrix from Example 2 is 2. The rank of the matrix from Example 3 is 2.

Theorem: The Row rank = Total Number of Variable − Number of free variables.

Theorem: A consistent linear system has a parametric solution if and only if the rank is less than the number of variables.

Reconsider Example 3. The rank of the matrix is 2, but there are three variables. We can also see from the reduced echelon form that the system is consistent. This theorem tells us, then, that there are infinite solutions that can be written in parametric form.

HOMOGENEOUS SYSTEMS

Definition: A **homogeneous system** is a system in which all of the constant terms are 0.

Example:

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 0 \\ 3x_1 + x_2 + 11x_3 = 0 \end{cases}$$

A homogeneous system is always consistent. It will always at least have the 0 solutions, or the trivial solution.

Theorem: If a homogeneous system has n variables and its coefficient matrix has rank r, then there are $n - r$ free variables (and thus the solution will have $n - r$ parameters)

Theorem: A homogeneous system with more unknowns than equations will ALWAYS have infinitely many solutions.

Example 4:

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 3x_4 = 0 \\ 2x_1 + 4x_2 + 3x_3 + 7x_4 = 0 \\ x_1 + 2x_2 + x_3 + x_4 = 0 \end{cases}$$

We know immediately from the previous theorem that this system will have infinite solutions because there are 4 variables but only 3 equations.

$$\begin{bmatrix} 1 & 2 & 2 & 3 & | & 0 \\ 2 & 4 & 3 & 7 & | & 0 \\ 1 & 2 & 1 & 1 & | & 0 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 2 & 3 & | & 0 \\ 0 & 0 & -1 & 1 & | & 0 \\ 0 & 0 & -1 & -2 & | & 0 \end{bmatrix} \xrightarrow[R_3+R_2]{R_2/2} \begin{bmatrix} 1 & 2 & 2 & 3 & | & 0 \\ 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & -3 & | & 0 \end{bmatrix} \xrightarrow[R_3/-3]{R_1-2R_2}$$

$$\begin{bmatrix} 1 & 2 & 0 & 5 & | & 0 \\ 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix} \xrightarrow[R_2+R_3]{R_1-5R_3} \begin{bmatrix} 1 & 2 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

This matrix has rank 3. There are 4 variables, so we should have $4 - 3 = 1$ paramater. The second column is not a pivot, so our parameter is x_2. This gives us the solution $x_1 = -2t, x_2 = t, x_3 = 0, x_4 = 0$.

NON-SQUARE SYSTEMS - systems in which the number of equations does not equal the number of unknowns

<u>CASE I</u>: The system has fewer equations than variables

Example:

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 4 \\ 2x_1 + x_2 - x_3 = 1 \end{cases}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & 4 \\ 2 & 1 & -2 & | & 1 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 3 & -2 & | & 4 \\ 0 & -5 & 2 & | & -7 \end{bmatrix} \xrightarrow{R_2/-5} \begin{bmatrix} 1 & 3 & -2 & | & 4 \\ 0 & 1 & -2/5 & | & -7/5 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -4/5 & | & 41/5 \\ 0 & 1 & -2/5 & | & -7/5 \end{bmatrix}$$

We can see that this system has infinite solutions:

$$\begin{cases} x = 41/5 \ + \ 4/5t \\ y = -7/5 + 2/5t \\ z = t \end{cases}$$

In this case, it is IMPOSSIBLE for the system to have a unique solution. If it is consistent, there will be at least one free variable because there aren't enough rows for every column to have a pivot.

Theorem: A system with n variables and p equations will have at least $n - p$ free variables.

<u>CASE II</u>: The system has more equations than variables

Example:

$$\begin{cases} 2x + 2y + z = 2 \\ 2x + y + 2z = 2 \\ 2x + y - z = 1 \\ 3x + 4y - z = 4 \end{cases}$$

$$\begin{bmatrix} 2 & 2 & 1 & | & 2 \\ 2 & 1 & 2 & | & 2 \\ 2 & 1 & -1 & | & 1 \\ 3 & 4 & -1 & | & 4 \end{bmatrix} \xrightarrow{R_4-R_1} \begin{bmatrix} 2 & 2 & 1 & | & 2 \\ 2 & 1 & 2 & | & 2 \\ 2 & 1 & -1 & | & 1 \\ 1 & 2 & -2 & | & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & -2 & | & 2 \\ 2 & 1 & 2 & | & 2 \\ 2 & 1 & -1 & | & 1 \\ 2 & 2 & 1 & | & 2 \end{bmatrix} \xrightarrow[R_4-2R_1]{R_2-2R_1,\ R_3-2R_1} \begin{bmatrix} 1 & 2 & -2 & | & 2 \\ 0 & -3 & 6 & | & -2 \\ 0 & -3 & 3 & | & -3 \\ 0 & -2 & 5 & | & -2 \end{bmatrix} \xrightarrow{R_3/-3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & -3 & 6 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 5 & -2 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 6 & -2 \\ 0 & -2 & 5 & -2 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - 2R_2} \\ \xrightarrow{R_3 + 3R_2} \\ \xrightarrow{R_4 + 2R_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow{R_3 - R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

We have a contradiction, so we can see that this system has no solutions.

Note: We can't say anything special about the number of solutions in this case.

Applications of Linear Systems of Equations

The Sudsy Detergent Company makes the laundry products Surf, Cyclone, Washo and Brisk from the raw materials anionic surfactant (A.S.) for lifting, sodium phosphate (S.P.) for water softening, and sodium sulphate (S.S.) for anti-lumping. The products are made in large batches, and the batches differ in size and proportions of raw materials because of the different manufacturing processes and equipment used for the products. The amounts of the raw materials used to make one batch of each product and the total amounts on hand are given in the table below. The amounts are given in 100-pound units.

	Surf	Cyclone	Washo	Brisk	On Hand
A.S.	4	8	4	4	60
S.P.	2	5	2	3	36
S.S.	3	7	4	3	50

Find the possible number of batches of each product that can be made such that all of the available raw materials are used. Define your unknowns here (what are you asked to find?)

$x_1 =$ ____

$x_2 =$ ____

$x_3 =$ ____

$x_4 =$ ____

Set up the system of equations here:

The row reduced echelon form matrix is here: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$

Now we just have to figure out what to do with these solutions.