MATRIX OPERATIONS

Dimensions of a Matrix: A matrix A with m rows and n columns is called an $m \times n$ matrix, and can be denoted by $A_{m\times n}$.

To index matrix entries, we use the notation $A = [a_{i,j}]$, where $a_{i,j}$ is the entry in the i-th row and j-th column of the matrix A.

Example: $A = \begin{bmatrix} 1 & 6 & -2 & 4 \\ 3 & 7 & -9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ -1 & 8 \\ 3 & 6 \end{bmatrix}$

A is a 2×4 matrix, $a_{2,3} = -9$, and $a_{1,2} = 6$.
B is a 3×2 matrix, $b_{3,2} = 6$, and $b_{1,2} = 7$.

Size: Two matrices A and B are said to have the same size or dimension if and only if they have the same number of rows and columns.

Equality: Two matrices are considered equal if they have the same dimension(or size) and their corresponding entries are equal (i.e. $a_{ij} = b_{ij}$ for every i and j).

Example: $\begin{bmatrix} a & 4 \\ 3 & b-1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$ if and only if $a = 6$ and $b - 1 = 2$. So $a = 6$ and $b = 3$.

Matrix Addition: If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same dimensions then $A + B = [a_{ij} + b_{ij}]$.

Scalar Multiplication: If $A = [a_{ij}]$ and k is a scalar (or a constant), then $kA = [ka_{ij}]$.

Example: If $A = \begin{bmatrix} 3 & -5 & -2 \\ 7 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ then

$A + B = \begin{bmatrix} 3 & -5 & -2 \\ 7 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & -7 & 4 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -12 & 2 \\ 9 & 4 & 2 \end{bmatrix}$ and

$$2A - B = 2\begin{bmatrix} 3 & -5 & -2 \\ 7 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & -7 & 4 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -10 & -4 \\ 14 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 7 & -4 \\ -2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -10 \\ 12 & -4 & 1 \end{bmatrix}$$

Properties of Matrices If a and b are scalars and A, B, and C are matrices, then

(i) $A + B = B + A$
(ii) $A + (B + C) = (A + B) + C$
(iii) $a(bA) = (ab)A$
(iv) $(a + b)A = aA + bA$
(v) $a(A + B) = aA + aB$
(vi) $A + \mathbf{0} = A$ and $a\mathbf{0} = \mathbf{0}$ (Note: **0** is the matrix with all 0 entries, called the zero matrix)

Matrix Multiplication: If $A = [a_{ij}]$ is $m \times n$ and $B = [b_{ij}]$ is $n \times r$, then AB is the $m \times r$ matrix $AB = [c_{ij}]$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Basically, this means you multiply the rows of A by the columns of B. More specifically, to get the (i, j)-th entry is the product of the corresponding values in row i of A with the values in column j of B and sum them together. - if you multiply the values of row i from A by the values from column j in B, the resulting value will be in row i, column j of AB)

Example: If $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 2 & 7 \end{bmatrix}$ then find AB

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} ((1)(4) + (2)(0) + (4)(2)) & ((1)(1) + (2)(-1) + (4)(7)) \\ ((2)(4) + (6)(0) + (0)(2)) & ((2)(1) + (6)(-1) + (0)(7)) \end{bmatrix} = \begin{bmatrix} 12 & 27 \\ 8 & -4 \end{bmatrix}$$

This is the result of the following calculations:
$(1)(4) + (2)(0) + (4)(2) = 12$
$(1)(1) + (2)(-1) + (4)(7) = 27$
$(2)(4) + (6)(0) + (0)(2) = 8$
$(2)(1) + (6)(-1) + (0)(7) = -4$

Properties of Matrix Multiplication Let a and b be scalars A, B, and C be matrices such that the product is defined. Then

(i) $A(BC) = (AB)C$
(ii) $(A + B)C = AC + BC$
(iii) $A(B + C) = AB + AC$
(iv) $\mathbf{0}A = A\mathbf{0} = \mathbf{0}$
(v) $(aA)(bB) = (ab)(AB)$

Note: The order in which you multiply matrices is VERY important. It is NOT always true that $AB = BA$ (but it does occasionaly happen). Sometimes, it is possible for AB to be defined when BA is not defined.
If $AB = AC$, it is not necessarily true that $B = C$.
Also, if $AD = 0$, it is possible that neither A nor D equal $\mathbf{0}$.

Identity Matrix: We've seen addition, scalar multiplication, and multiplication. We've also seen the matrix version of 0. The matrix version of 1 is called the **identity matrix**, and is written I.

$$I = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ : & & 1 & & : \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

I is called the *multiplicative identity* for matrices since $AI = IA = A$.

Matrix Transpose: The transpose of a matrix A is the matrix A^T where the rows and columns of A have been interchanged.

$$A = [a_{ij}], A^T = [a_{ji}]$$

Note that there is no analogy with numbers.

Example: If $A = \begin{bmatrix} 3 & 4 & -2 & 0 \\ 8 & -9 & 12 & 1 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & 8 \\ 4 & -9 \\ -2 & 12 \\ 0 & 1 \end{bmatrix}$

Note: $(A+B)^T = A^T + B^T$, $(A^T)^T = A$, and $(AB)^T = B^T A^T$

Matrix Equations: It is easy to solve matrix equations like $2A + X = B$, but solving $AX = B$ is a little more difficult. We're going to see how we can actually solve systems of equations by setting up matrix equations.

Given

$$\begin{cases} 4x - 2y + 5z = 4 \\ 5x + 3y - 2z = 5 \\ 7x - y - 3z = 1 \end{cases}$$

we can set up the matrix equation $AX = B$ as follows: $\begin{bmatrix} 4 & -2 & 5 \\ 5 & 3 & -2 \\ 7 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$

If we could find A^{-1}, then it would be a simple matter to solve the above equation. However, it is important to note that not all matrices have inverses.

INVERSES

Inverses of Matrices:
A square matrix A is said to be **invertible** if there exists a matrix B such that $BA = I$ and $AB = I$, where I is the identity matrix. B is called the inverse of A and is often written A^{-1}.

If A has no inverse, it is called **singular**.

Theorem: Let A and B be invertible square matrices. Then

(i.) A^{-1} is unique. (there is only one inverse)
(ii.) A^{-1} is invertible, and $(A^{-1})^{-1} = A$
(iii.) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$
(iv.) A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, ...$
(v.) kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

Solving for the Inverse:
Step 1: Set up the augmented matrix $[A|I]$.
Step 2: Perform elementary row operations on A in order to get it to reduced row echelon form.
Step 3: These operations will turn I into A^{-1}. You will end with the augmented matrix $[I|A^{-1}]$
Step 4: If it is impossible to reduce A to I, then A is not invertible.

Shortcut for 2×2 matrices: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solving Matrix Equations using the Inverse: Consider the equation $AX = B$:

(1) Multiply both sides on the left by A^{-1} to get $A^{-1}AX = A^{-1}B$
(2) Since by definition $A^{-1}A = I$, we get $IX = A^{-1}B$
(3) I is the multiplicative identity, so the left side simplifies giving $X = A^{-1}B$.

Example: Solve the equation $AX = B$ for X when $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$

Solution: Step 1: Solve for A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 2R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 - 3R_3]{R_1 + 9R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \xrightarrow{R_3/(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Step 2: Multiply both sides of the equation on the left by A^{-1} to get $X = A^{-1}B$

$$X = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} (-40)(5) + (16)(3) + (9)(17) \\ (13)(5) + (-5)(3) + (3)(17) \\ (5)(5) + (-2)(3) + (-1)(17) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

So we get $x = 1$, $y = -1$, and $z = 2$.

Equivalent Statements: Let A be an $n \times n$ matrix, then the following are equivalent (if one is true, they all are)
1. A is invertible
2. $AX = 0$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. $AX = B$ is consistent for every $n \times 1$ matrix B
5. $AX = B$ has a unique solution for every nx1 matrix B ($X = A^{-1}B$).

Special Matrices

Upper Triangular Matrix — a square matrix in which all the entries below the main diagonal are 0, e.g. $\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

Lower Triangular Matrix — a square matrix in which all the entries below the main diagonal are 0, e.g. $\begin{bmatrix} 7 & 0 & 0 \\ 1 & -9 & 0 \\ 3 & 2 & 8 \end{bmatrix}$

A matrix is **triangular** if it is either upper triangular or lower triangular.

Diagonal Matrix —- a square matrix in which all entries NOT on the main diagonal are 0, e.g. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (Note: every diagonal matrix is upper and lower triangular)

THEOREM: A diagonal matrix is invertible if and only if its diagonal entries are all non-zero.

In the example given of the diagonal matrix, all of the diagonal entries are non-zero, so this theorem tells us that it must be invertible.

Symmetric - a square matrix is symmetric if $A = A^T$: $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Notice that the numbers above and below the diagonal are the same.

PROPERTIES OF SYMMETRIC MATRICES – if A and B are symmetric matrices of the same size and k is any scalar then
1. A^T is symmetric
2. $A + B$ and $A - B$ are symmetric
3. kA is symmetric
4. AB is symmetric if and only if $AB = BA$
5. If A is invertible, then A^{-1} is symmetric

Matrix Operations Properties

Properties of Matrices If a and b are scalars and A, B, and C are matrices, then

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(iii) $a(bA) = (ab)A$
(iv) $(a + b)A = aA + bA$
(v) $a(A + B) = aA + aB$
(vi) $A + \mathbf{0} = A$ and $A\mathbf{0} = \mathbf{0}$
Note: $\mathbf{0}$ is a matrix with all 0 entries.

Properties of the Transpose, A^T: Let a be a scalar and A and B be $n \times n$ square matrices. Then

a) $(A^T)^T = A$
b) $(A + B)^T = A^T + B^T$
c) $(aA)^T = aA^T$
d) $(AB)^T = B^T A^T$

Properties of Matrix Multiplication Let a and b be scalars A, B, and C be matrices such that the product is defined. Then

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Properties of Inverses: Let A and B be invertible square matrices. Then

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